A set $D$ of vertices of a graph $G$ is a total efficient dominating set if every vertex in $V$ is adjacent to exactly one vertex in $D$. The total efficient domination number $\gamma_{te}(G)$ of $G$ is the minimum cardinality of a total efficient dominating set of $G$. In this paper, the exact values of $\gamma_{te}(G)$ for some standard graphs are found and some bounds are obtained. Also a Nordhaus-Gaddum type result is established. In addition, the total efficient domatic number $d_{te}(G)$ of $G$ is defined to be maximum order of a partition of the vertex set of $G$ into total efficient dominating sets of $G$. Also a relation between $\gamma_{te}(G)$ and $d_{te}(G)$ is established.

**Keywords:** efficient dominating set, total dominating set, total efficient dominating set, total efficient domination number.

**Mathematics Subject Classification:** 05C.

1. **INTRODUCTION**

By a graph, we mean a finite, undirected without loops, multiple edges and isolated vertices. Terms not defined here may be found in Kulli [1].

A set $D$ of vertices in a graph $G=(V, E)$ is called a dominating set if every vertex in $V-D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Recently many new domination parameters are given in the book by Kulli [2, 3, 4].

A dominating set $D$ of $G$ is an efficient dominating set if every vertex in $V-D$ is adjacent to exactly one vertex in $D$. The efficient domination number $\gamma_e(G)$ of $G$ is the minimum cardinality of an efficient dominating set of $G$. This concept was studied, for example, in [5, 6, 7]. Many other domination parameters in domination theory were studied, for example, in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

A set $D$ of vertices in a graph $G$ is a total dominating set if every vertex in $V$ is adjacent to some vertex in $D$. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality of a total dominating set of $G$.

In [28], Kulli and Patwari introduced the concept total efficient domination as follows:

A set $D$ of vertices in a graph $G$ is a total efficient dominating set of $G$ if every vertex in $V-D$ is adjacent to exactly one vertex in $D$. The total efficient domination number $\gamma_{te}(G)$ of $G$ is the minimum cardinality of a total efficient dominating set of $G$.

A $\gamma_e$-set is a minimum total efficient dominating set. Let $\Delta(G)$ ($\delta(G)$) denote the maximum (minimum) degree among the vertices of $G$. Let $\lceil x \rceil$ denote the least integer greater than or equal to $x$.

We note that $\gamma_t(G)$ and $\gamma_{te}(G)$ are only defined for $G$ with $\delta(G) \geq 1$. 

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2. TOTAL EFFICIENT DOMINATION NUMBER

We list the exact values of the total efficient domination number for some standard graphs.

**Proposition 1:** If $P_p$ is a path with $p$ vertices, then
\[
\gamma_{te}(P_p) = \left\lfloor \frac{p}{2} \right\rfloor, \quad \text{when } p = 0 \pmod{4} \text{ and } p = 3 \pmod{4}.
\]

**Proposition 2:** If $C_p$ is a cycle with $p$ vertices, then
\[
\gamma_{te}(C_p) = \left\lfloor \frac{p}{2} \right\rfloor, \quad \text{when } p = 0 \pmod{4}.
\]

**Proposition 3:** If $K_{m,n}$ is a complete bipartite graph, $1 \leq m \leq n$, then
\[
\gamma_{te}(K_{m,n}) = 2.
\]

**Remark 4:** Every graph $G$ without isolated vertices does not contain a total efficient dominating set. It implies that $\gamma_{te}(G)$ does not exist. For example, if $u$, $v$, $w$ are three cutvertices of a tree $T$ such that $\deg u \geq 3$, $\deg v \geq 3$, $\deg w \geq 3$ and $uv$ and $vw$ are edges of $T$, then $\gamma_{te}(T)$ does not exist.

**Proposition 5:** If $K_p$ is a complete graph with $p \geq 3$ vertices, then $\gamma_{te}(K_p)$ does not exist.

**Proposition 6:** If $\gamma_{te}(G)$ exists, then
\[
\gamma(G) \leq \gamma_{te}(G)
\]
and this bound is sharp.

**Proof:** Clearly every total efficient dominating set is a total dominating set. Thus (1) holds.

The complete bipartite graphs $K_{m,n}$, $1 \leq m \leq n$ achieve this bound.

**Proposition 7:** If $\gamma_{te}(G)$ exists, then
\[
\gamma(G) \leq \gamma_{te}(G)
\]
and this bound is sharp.

**Proof:** Clearly every efficient total dominating set is an efficient dominating set. Thus (2) holds.

The complete bipartite graphs $K_{m,n}$, $2 \leq m \leq n$ achieve this bound.

The following theorem gives an upper bound for $\gamma_{te}(G)$.

**Theorem 8:** For any graph $G$ without isolated vertices,
\[
\gamma_{te}(G) \leq p - \Delta(G) + 1
\]
and this bound is sharp.

**Proof:** Let $D$ be a $\gamma_{te}$-set of $G$.

Suppose $u \in V - D$. Then $\deg_G u$ is at most $|V - D|$ as it is adjacent to a vertex in $D$ and may be adjacent to every vertex of $V - D$ other than itself. Hence the maximum degree of a vertex in $V - D$ is $|V - D|$. Thus $|V - D| = p - \gamma_{te}(G)$.

Suppose $u \in D$. Then $\deg_G u$ is at most $|V - D| + 1$.

Thus
\[
\Delta(G) \leq |V - D| + 1.
\]

or
\[
\gamma_{te}(G) \leq p - \Delta(G) + 1.
\]

The graphs $mK_2$, $m \geq 1$ achieve this bound.

The following theorem gives a lower bound for $\gamma_{te}(G)$.
Theorem 9: Let $G$ be a $(p, q)$ connected graph with $p \geq 2$ vertices. Then

$$2(p - q) \leq \gamma_a(G).$$

Furthermore, equality holds if and only if $G$ is a tree with exactly one cutvertex or exactly two cutvertices.

Proof: Let $D$ be a $\gamma_a$-set of $G$. Then for each vertex $u \in V - D$, there exists a vertex $v$ in $D$ such that $uv \in E$. Also for each vertex $x \in D$, there exists unique vertex $y \in D$ such that $xy \in E$. Thus

$$q \geq \left\lfloor \frac{|D|}{2} + |V - D| \right\rfloor$$

or

$$2q \geq |D| + 2|V - D|$$

or

$$2q \geq \gamma_a(G) + 2p - 2\gamma_a(G)$$

or

$$2(q - p) \leq \gamma_a(G).$$

We prove the second part.

Suppose $G$ is a tree with exactly one cutvertex or two cutvertices. Then $\gamma_a(G) = 2 = 2(p - q)$, since $p - q = 1$.

Conversely suppose $\gamma_a(G) = 2(p - q)$. We now prove that $G$ is a tree with at most two cutvertices. Clearly for any graph without isolated vertices, $\gamma_a(G) \geq 2$.

Suppose $p < q$. Then $2(p - q)$ is negative, which is a contradiction.

Suppose $p = q$. Then $2(p - q)$ is zero, which is a contradiction.

Suppose $p > q$. Since $G$ is connected, it implies that $G$ is a tree. If $G$ is a tree with exactly 3 vertices, then by Remark 4, $\gamma_a(G)$ does not exist. If $G$ is a tree with at least 4 cutvertices, then $\gamma_a(G) \geq 4 \neq 2(p - q)$, since $p - q = 1$. Thus we conclude that $G$ is a tree with at most two cutvertices.

Next we characterize graphs for which $\gamma_a(G) = p$.

Theorem 10: Let $G$ be graph without isolated vertices and with $p \geq 2$ vertices. Then $\gamma_a(G) = p$ if and if $G=mK_2$, $m\geq1$.

Proof: Suppose $G=mK_2$, $m\geq1$. Obviously $\gamma_a(G) = p$.

Conversely suppose $\gamma_a(G) = p$. We now prove that $G=mK_2$, $m\geq1$. Assume $G\neq mK_2$. Then $\deg_G u \geq 2$. Let $D$ be a $\gamma_a$-set of $G$. Since $\gamma_a(G) = p$, it implies that $|V - D| = \emptyset$. Hence $u \notin D$. Since $\deg_G u \geq 2$, it implies that $u$ is adjacent with at least two vertices in $D$, which is a contradiction. Suppose $\deg_G u < 1$. Then $u$ is an isolated vertex, again a contradiction. Thus $\deg_G u = 1$. Since $u$ is arbitrary, it follows that $G=mK_2$, $m\geq1$.

The following theorem gives a lower bound for $\gamma_a(T)$.

Theorem 11: Let $T$ be a tree with $p \geq 3$ vertices, If $\gamma_a(T)$ exists, then

$$\gamma_a(T) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$$

where $m$ is the number of cutvertex of $T$.

Proof: Let $T$ be a tree with $p \geq 3$ vertices. Suppose $\gamma_a(T)$ exists. We now prove that $\gamma_a(T) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$. On the contrary, assume $\gamma_a(T) > \left\lfloor \frac{m}{2} \right\rfloor + 1$. Then there exist 3 cutvertices $u, v, w$ in $D$ such that $uv, vw$ are edges of $T$ where $D$ is a $\gamma_a$-set of $T$. By Remark 4, $\gamma_a(T)$ does not exist, which is a contradiction. This prove that $\gamma_a(T) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$.

We obtain a relation between the total efficient domination number $\gamma_a(G)$ and the chromatic number $\chi(G)$. Relations between some parameters and the chromatic number established in [29].

We need the following result.

Theorem 12/[2, p.8]: For any graph $G$, $\chi(G) \leq \Delta(G) + 1$.  

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Theorem 13: For any graph $G$ without isolated vertices,
\[ \gamma_{te}(G) + \chi(G) \leq p + 2 \]  
and this bound is sharp.

**Proof:** By Theorem 8, $\gamma_{te}(G) \leq p - \Delta(G) + 1$ and by Theorem 12, $\chi(G) \leq \Delta(G) + 1$. Thus (3) holds.

The graphs $mK_2, m \geq 1$ achieve this bound.

Nordhaus-Gaddum type results were obtained for many parameters, for example, in [30, 31, 32, 33, 34, 35, 36].

We now establish Nordhaus-Gaddum type result.

**Theorem 14:** Let $G$ and $\overline{G}$ have no isolated vertices. If both $\gamma_{te}(G)$ and $\gamma_{te}(\overline{G})$ exist, then
\[ 4 \leq \gamma_{te}(G) + \gamma_{te}(\overline{G}) \leq p + 3. \]

**Proof:** Let $G$ and $\overline{G}$ have no isolated vertices. If both $\gamma_{te}(G)$ and $\gamma_{te}(\overline{G})$ exist, then $\gamma_{te}(G) \geq 2$ and $\gamma_{te}(\overline{G}) \geq 2$.

Therefore
\[ 4 \leq \gamma_{te}(G) + \gamma_{te}(\overline{G}). \]

By Theorem 8, we have
\[ \gamma_{te}(G) \leq p - \Delta(G) + 1. \]

Therefore
\[ \gamma_{te}(G) \leq p - \delta(G) + 1. \]

Also we have
\[ \gamma_{te}(\overline{G}) \leq p - \Delta(\overline{G}) + 1. \]

Thus
\[ \gamma_{te}(G) + \gamma_{te}(\overline{G}) \leq 2p - \left[ \delta(G) + \Delta(\overline{G}) \right] + 2 \]
\[ \leq p - (p - 1) + 2 \]
\[ \leq p + 3. \]

The graph $P_4$ achieves the lower bound.

### 3. TOTAL EFFICIENT DOMATIC NUMBER

**Definition 15:** The total efficient domatic number $d_{te}(G)$ of a graph $G$ is the maximum order of a partition of the vertex set of $G$ into total efficient dominating sets of $G$.

We obtain the exact values of the total efficient domatic number $d_{te}(G)$ for some standard graphs.

**Proposition 16:** For any cycle $C_{4n}, n \geq 1$,
\[ d_{te}(C_{4n}) = 2. \]

**Proposition 17:** For any complete bipartite graph $K_{m,n}, 1 \leq m \leq n$,
\[ d_{te}(K_{m,n}) = m. \]

**Proposition 18:** For any tree $T$ with $p \geq 2$ vertices,
\[ d_{te}(T) = 1. \]

**Proposition 19:** Let $G$ be a graph without isolated vertices. If $\gamma_{te}(G)$ exists, then
\[ d_{te}(G) \leq \frac{p}{\gamma_{te}(G)}. \]

**Proposition 20:** Let $G$ be a graph without isolated vertices. If $d_{te}(G)$ exists, then
\[ d_{te}(G) \leq \delta(G). \]
Proposition 21: If $G$ is a graph without isolated vertices and if $\gamma_t(G)$ exists, then
$$\gamma_t(G) + d_t(G) \leq p + 1.$$  
Furthermore, equality holds if $G = mK_2$, $m \geq 1$.

Proof: By Theorem 6, we have
$$\gamma_t(G) \leq p - \Delta(G) + 1$$
or
$$\gamma_t(G) \leq p - \delta(G) + 1.$$  
By Proposition 20, we have
$$\gamma_t(G) \leq \delta(G).$$  
Hence
$$\gamma_t(G) + d_t(G) \leq p + 1.$$  
We prove the second part.

If $G = mK_2$, $m \geq 1$ then by Theorem 10, $\gamma_t(G) = p$. Also $d_t(G) = 1$. Thus $\gamma_t(G) + d_t(G) = p + 1$.

4. SOME OPEN PROBLEMS

Problem 1: Characterize graphs $G$ for which $\gamma_t(G) = \gamma_e(G)$.

Problem 2: Characterize graphs $G$ for which $\gamma(G) = \gamma_e(G)$.

Problem 3: Characterize graphs $G$ for which $\gamma_t(G) = p - \Delta(G) + 1$.

Problem 4: Characterize trees $T$ for which $\gamma_t(T) = \left\lceil \frac{m}{2} \right\rceil + 1$ where $m$ is the number of cutvertices of $T$.

Problem 1: Characterize graphs $G$ for which $\gamma_t(G) + d_t(G) = p + 1$.

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