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TOTAL EFFICIENT DOMINATION IN GRAPHS

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ABSTRACT

A set D of vertices of a graph G is a total efficient dominating set if every vertex in V is adjacent to exactly one vertex in D. The total efficient domination number $\gamma_{te}(G)$ of G is the minimum cardinality of a total efficient dominating set of G. In this paper, the exact values of $\gamma_{te}(G)$ for some standard graphs are found and some bounds are obtained. Also a Nordhaus-Gaddum type result is established. In addition, the total efficient domatic number $d_{te}(G)$ of G is defined to be maximum order of a partition of the vertex set of G into total efficient dominating sets of G. Also a relation between $\gamma_{te}(G)$ and $d_{te}(G)$ is established.

Keywords: *efficient dominating set, total dominating set, total efficient dominating set, total efficient domination number.*

Mathematics Subject Classification: *05C.*

1. INTRODUCTION

By a graph, we mean a finite, undirected without loops, multiple edges and isolated vertices. Terms not defined here may be found in Kulli [1].

A set D of vertices in a graph $G=(V, E)$ is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many new domination parameters are given in the book by Kulli [2, 3, 4].

A dominating set D of G is an efficient dominating set if every vertex in $V - D$ is adjacent to exactly one vertex in D. The efficient domination number $\gamma_e(G)$ of G is the minimum cardinality of an efficient dominating set of G. This concept was studied, for example, in [5, 6, 7]. Many other domination parameters in domination theory were studied, for example, in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

A set D of vertices in a graph G is a total dominating set if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

In [28], Kulli and Patwari introduced the concept total efficient domination as follows:

A set D of vertices in a graph G is a total efficient dominating set of G if every vertex in V is adjacent to exactly one vertex in D. The total efficient domination number $\gamma_{te}(G)$ of G is the minimum cardinality of a total efficient dominating set of G.

A γ_{te}-set is a minimum total efficient dominating set. Let $\Delta(G)$ ($\delta(G)$) denote the maximum (minimum) degree among the vertices of G. Let $\lceil x \rceil$ denote the least integer greater than or equal to x.

We note that $\gamma_t(G)$ and $\gamma_{te}(G)$ are only defined for G with $\delta(G) \geq 1$.

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2. TOTAL EFFICIENT DOMINATION NUMBER

We list the exact values of the total efficient domination number for some standard graphs.

Proposition 1: If P_p is a path with p vertices, then

$$\gamma_{te}\left(P_p\right)=\left\lceil\frac{p}{2}\right\rceil, \text{ when } p = 0 \text{ (mod 4) and } p = 3 \text{ (mod 4).}$$

Proposition 2: If C_p is a cycle with p vertices, then

$$\gamma_{te}\left(C_p\right)=\frac{p}{2}, \text{ when } p = 0 \text{ (mod 4).}$$

Proposition 3: If $K_{m,n}$ is a complete bipartite graph, $1\leq m\leq n$, then

$$\gamma_{te}\left(K_{m,n}\right)=2.$$

Remark 4: Every graph G without isolated vertices does not contain a total efficient dominating set. It implies that $\gamma_{te}(G)$ does not exist. For example, if u, v, w are three cutvertices of a tree T such that deg $u\geq 3$, deg $v \geq 3$, deg $w \geq 3$ and uv and vw are edges of T, then $\gamma_{te}(T)$ does not exist.

Proposition 5: If K_p is a complete graph with $p\geq 3$ vertices, then $\gamma_{te}(K_p)$ does not exist.

Proposition 6: If $\gamma_{te}(G)$ exists, then

$$\gamma_t(G) \leq \gamma_{te}(G) \tag{1}$$

and this bound is sharp.

Proof: Clearly every total efficient dominating set is a total dominating set. Thus (1) holds.

The complete bipartite graphs $K_{m,n}$, $1\leq m \leq n$ achieve this bound.

Proposition 7: If $\gamma_{te}(G)$ exists, then

$$\gamma_e(G) \leq \gamma_{te}(G) \tag{2}$$

and this bound is sharp.

Proof: Clearly every efficient total dominating set is an efficient dominating set. Thus (2) holds.

The complete bipartite graphs $K_{m,n}$, $2\leq m \leq n$ achieve this bound.

The following theorem gives an upper bound for $\gamma_{te}(G)$.

Theorem 8: For any graph G without isolated vertices,

$$\gamma_{te}(G) \leq p - \Delta(G) + 1$$

and this bound is sharp.

Proof: Let D be a γ_{te}-set of G.

Suppose $u \in V - D$. Then $\deg_G u$ is at most $|V - D|$ as it is adjacent to a vertex in D and may be adjacent to every vertex of $V - D$ other than itself. Hence the maximum degree of a vertex in $V - D$ is $|V - D|$. Thus $|V - D| = p - \gamma_{te}(G)$.

Suppose $u \in D$. Then $\deg_G u$ is at most $|V - D|+1$.

Thus
$$\begin{aligned}\Delta(G) &\leq |V - D|+1.\\ &\leq p - \gamma_{te}(G) + 1\end{aligned}$$

or
$$\gamma_{te}(G) \leq p - \Delta(G) + 1.$$

The graphs mK_2, m≥1 achieve this bound.

The following theorem gives a lower bound for $\gamma_{te}(G)$.

Theorem 9: Let G be a (p, q) connected graph with $p \geq 2$ vertices. Then

$$2(p - q) \leq \gamma_{te}(G).$$

Furthermore, equality holds if and only if G is a tree with exactly one cutvertex or exactly two cutvertices.

Proof: Let D be a γ_{te}-set of G. Then for each vertex $u \in V - D$, there exists a vertex v in D such that $uv \in E$. Also for each vertex $x \in D$, there exists unique vertex $y \in D$ such that $xy \in E$. Thus

$$q \geq \frac{|D|}{2} + |V - D|$$

or $$2q \geq |D| + 2|V - D|$$

or $$2q \geq \gamma_{te}(G) + 2p - 2\gamma_{te}(G)$$

or $$2(q - p) \leq \gamma_{te}(G).$$

We prove the second part.

Suppose G is a tree with exactly one cutvertex or two cutvertices. Then $\gamma_{te}(G) = 2 = 2(p - q)$, since $p - q = 1$.

Conversely suppose $\gamma_{te}(G) = 2(p - q)$. We now prove that G is a tree with at most two cutvertices. Clearly for any graph without isolated vertices, $\gamma_{te}(G) \geq 2$.

Suppose $p < q$. Then $2(p - q)$ is negative, which is a contradiction.

Suppose $p = q$. Then $2(p - q)$ is zero, which is a contradiction.

Suppose $p > q$. Since G is connected, it implies that G is a tree. If G is a tree with exactly 3 vertices, then by Remark 4, $\gamma_{te}(G)$ does not exist. If G is a tree with at least 4 cutvertices, then $\gamma_{te}(G) \geq 4 \neq 2(p - q)$, since $p - q=1$. Thus we onclude that G is a tree with at most two cutvertices.

Next we characterize graphs for which $\gamma_{te}(G) = p$.

Theorem 10: Let G be graph without isolated vertices and with $p \geq 2$ vertices. Then $\gamma_{te}(G) = p$ if and if $G=mK_2$, $m\geq1$.

Proof: Suppose $G=mK_2$, $m\geq1$. Obviously $\gamma_{te}(G) = p$.

Conversely suppose $\gamma_{te}(G) = p$. We now prove that $G=mK_2$, $m\geq1$. Assume $G\neq mK_2$. Then $\deg_G u \geq 2$. Let D be a γ_{te}-set of G. Since $\gamma_{te}(G) = p$, it implies that $|V - D| = \phi$. Hence $u\in D$. Since $\deg_G u \geq 2$, it implies that u is adjacent with at least two vertices in D, which is a contradiction. Suppose $\deg_G u < 1$. Then u is an isolated vertex, again a contradiction. Thus $\deg_G u = 1$. Since u is arbitrary, it follows that $G=mK_2$, $m\geq1$.

The following theorem gives a lower bound for $\gamma_{te}(T)$.

Theorem 11: Let T be a tree with $p \geq 3$ vertices, If $\gamma_{te}(T)$ exists, then

$$\gamma_{te}(T) \leq \left\lceil \frac{m}{2} \right\rceil + 1$$

where m is the number of cutvertex of T.

Proof: Let T be a tree with $p\geq3$ vertices. Suppose $\gamma_{te}(T)$ exists. We now prove that $\gamma_{te}(T) \leq \left\lceil \frac{m}{2} \right\rceil + 1.$ On the contrary, assume $\gamma_{te}(T) > \left\lceil \frac{m}{2} \right\rceil + 1.$ Then there exist 3 cutvertices u, v, w in D such that uv, vw are edges of T where D is a γ_{te}-set of T. By Remark 4, $\gamma_{te}(T)$ does not exist, which is a contradiction. This prove that $\gamma_{te}(T) \leq \left\lceil \frac{m}{2} \right\rceil + 1.$

We obtain a relation between the total efficient domination number $\gamma_{te}(G)$ and the chromatic number $\chi(G)$. Relations between some parameters and the chromatic number established in [29].

We need the following result.

Theorem 12[2, *p*.8]: For any graph G, $\chi(G) \leq \Delta(G)+1$.

Theorem 13: For any graph G without isolated vertices,

$$\gamma_{te}(G) + \chi(G) \leq p + 2 \tag{3}$$

and this bound is sharp.

Proof: By Theorem 8, $\gamma_{te}(G) \leq p - \Delta(G) + 1$ and by Theorem 12, $\chi(G) \leq \Delta(G)+1$. Thus (3) holds.

The graphs mK_2, $m\geq 1$ achieve this bound.

Nordhaus-Gaddum type results were obtained for many parameters, for example, in [30, 31, 32, 33, 34, 35, 36].

We now establish Nordhaus-Gaddum type result.

Theorem 14: Let G and $\overline{G}$ have no isolated vertices. If both $\gamma_{te}(G)$ and $\gamma_{te}(\overline{G})$ exist, then

$$4 \leq \gamma_{te}(G) + \gamma_{te}(\overline{G}) \leq p + 3.$$

Proof: Let G and $\overline{G}$ have no isolated vertices. If both $\gamma_{te}(G)$ and $\gamma_{te}(\overline{G})$ exist, then $\gamma_{te}(G) \geq 2$ and $\gamma_{te}(\overline{G}) \geq 2$. Therefore

$$4 \leq \gamma_{te}(G) + \gamma_{te}(\overline{G}).$$

By Theorem 8, we have

$$\gamma_{te}(G) \leq p - \Delta(G) + 1.$$

Therefore $\quad \gamma_{te}(G) \leq p - \delta(G) + 1.$

Also we have

$$\gamma_{te}(\overline{G}) \leq p - \Delta(\overline{G}) + 1.$$

Thus

$$\begin{aligned} \gamma_{te}(G) + \gamma_{te}(\overline{G}) &\leq 2p - \left[\delta(G) + \Delta(\overline{G})\right] + 2 \\ &\leq p - (p-1) + 2 \\ &\leq p + 3. \end{aligned}$$

The graph P_4 achieves the lower bound.

3. TOTAL EFFICIENT DOMATIC NUMBER

Definition 15: Toe total efficient domatic number $d_{te}(G)$ of a graph G is the maximum order of a partition of the vertex set of G into total efficient dominating sets of G.

We obtain the exact values of the total efficient domatic number $d_{te}(G)$ for some standard graphs.

Proposition 16: For any cycle C_{4n}, $n\geq 1$,

$$d_{te}(C_{4n}) = 2.$$

Proposition 17: For any complete bipartite graph $K_{m,n}$, $1\leq m\leq n$,

$$d_{te}(K_{m,n}) = m.$$

Proposition 18: For any tree T with $p\geq 2$ vertices,

$$d_{te}(T) = 1.$$

Proposition 19: Let G be a graph without isolated vertices. If $\gamma_{te}(G)$ exists, then

$$d_{te}(G) \leq \frac{p}{\gamma_{te}(G)}.$$

Proposition 20: Let G be a graph without isolated vertices. If $d_{te}(G)$ exists, then

$$d_{te}(G) \leq \delta(G).$$

Proposition 21: If G is a graph without isolated vertices and if $\gamma_{te}(G)$ exists, then

$$\gamma_{te}(G) + d_{te}(G) \leq p + 1.$$

Furthermore, equality holds if $G=mK_2$, $m \geq 1$.

Proof: By Theorem 6, we have

$$\gamma_{te}(G) \leq p - \Delta(G) + 1$$

or $$\gamma_{te}(G) \leq p - \delta(G) + 1.$$

By Proposition 20, we have $\gamma_{te}(G) \leq \delta(G)$.

Hence $$\gamma_{te}(G) + d_{te}(G) \leq p + 1.$$

We prove the second part.

If $G=mK_2$, $m \geq 1$ then by Theorem 10, $\gamma_{te}(G) = p$. Also $d_{te}(G)=1$. Thus $\gamma_{te}(G)+d_{te}(G)=p+1$.

4. SOME OPEN PROBLEMS

Problem 1: Characterize graphs G for which $\gamma_t(G) = \gamma_{te}(G)$.

Problem 2: Characterize graphs G for which $\gamma_e(G) = \gamma_{te}(G)$.

Problem 3: Characterize graphs G for which $\gamma_t(G) = p - \Delta(G) + 1$.

Problem 4: Characterize trees T for which $\gamma_{te}(T) = \left\lceil \frac{m}{2} \right\rceil + 1$ where m is the number of cutvertices of T.

Problem 1: Characterize graphs G for which $\gamma_{te}(G) + d_{te}(G) = p + 1$.

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