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ON $(E,1)(N,P_n)$ SUMMABILITY OF FOURIER SERIES

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ABSTRACT

In this paper we have establish theorem concerning $(E,1)(N,P_n)$ product summability of Fourier series.

Keywords: *(E,q) summability, (N,P_n) summability, $(E,1)(N,P_n)$ summability.*

1. DEFINITION AND NOTATION

Let $f(x)$ be a periodic function with period 2π and is integrable in Lebesgue sense over $(-\pi,\pi)$.

Let $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=1}^{\infty}A_n(x)$ (1.1)

be the Fourier series of $f(x)$. The series

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(x) \tag{1.2}$$

is called the conjugate series of the Fourier series.

An infinite series $\sum a_n$ with the sequence of partial sum $\{s_n\}$ is said to be $(E,1)(N,P_n)$ summable to s, if

$$K_n(t)=\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{p_k}\sum_{v=0}^{k}p_{k-v}\frac{\sin\left(v+\frac{1}{2}t\right)}{\sin\frac{t}{2}}\right\}\to s\text{, as } n\to\infty.$$

We shall use the following notation,

$$\phi(t)=f(x+t)+f(x-t)-2s$$

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$$

and $$K_n(t)=\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{p_k}\sum_{v=0}^{k}p_{k-v}\frac{\sin\left(v+\frac{1}{2}t\right)}{\sin\frac{t}{2}}\right\}$$

$\tau=\left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater that $\frac{1}{t}$

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2. INTRODUCTION

The product summability of various summability have been studied since 1919, but now it seems to be more study after 1990's. So many researchers like Mittal, M.L. and Prasad, G. [7], Chandra, P. [2], Varshney, O.P. [11], Dikshit, H.P. [3], Sahney, B.N. [9], Sinha, Santosh Kumar and Shrivastava, U.K. [10], Lal, Shyam and Nigam, Hare Krishna [5], Mohanty, R. and Nanda, M. [8] and many more gives the result on the product summability of Fourier series and its allied series.

Under a general condition, here we have proved a theorem on product summability $(E,1)(N,P_n)$ of Fourier series.

3. MAIN THEOREM

Theorem 1: Let $\{c_n\}$ be a non-negative, monotonic, non-increasing sequence of real constants such that

$$C_n = \sum_{v=1}^{n} c_n \to \infty \text{ as } n \to \infty$$

If
$$\Phi(t) = \int_0^t |\phi(u)|du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right)C_\tau}\right] \text{ as } t \to +0 \tag{3.1}$$

where, $\alpha(t)$ is positive, monotonic and non-increasing function of t and

$$\log(n+1) = O[\{\alpha(n+1)\}C_{n+1}], \text{ as } n \to \infty \tag{3.2}$$

then the Fourier series (1.1) is $(E,1)(N,P_n)$ summable to zero at point x.

4. LEMMAS

Lemma 1: For $0 \le t \le \frac{1}{n+1}$, $|K_n(t)| = O(n)$

Proof: We have, for $0 \le t \le \frac{1}{n+1}$

$$\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi} \quad \text{and } \sin nt \le n \sin t$$

$$|K_n(t)| = \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\frac{(2v+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}(2k+1)\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\right\}\right|$$

$$\le \frac{(2n+1)}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\right|$$

$$= \frac{(2n+1)}{2^{n+1}\pi}.2^n$$

$$= \frac{(2n+1)}{2\pi}$$
$$= O(n)$$

Lemma 2: For $\frac{1}{n+1} \le t \le \pi,\ |K_n(t)| = O\left(\frac{1}{t}\right)$

Proof: We have $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $\sin nt \le 1$

$$|K_n(t)| = \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$
$$\le \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\right\}\right|$$
$$\le \frac{1}{2^{n+1}t}\left|\sum_{k=0}^{n}\binom{n}{k}\right|$$
$$= \frac{1}{2^{n+1}t}.2^n$$
$$= O\left(\frac{1}{t}\right)$$

5. Proof of theorem1: Following Zygmund [12], the n^{th} partial sum of the Fourier series (1.1) can be written as

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The (N,P_n) transform of $s_n(x)$ is given by

$$t_n^{\ N} - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

The $(E,1)(N,P_n)$ transform of $s_n(x)$ is given by

$$t_n^{\ EN} - f(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt\right\}$$
$$= \int_0^{\pi}\phi(t).K_n(t)dt$$
$$= \int_0^{1/n+1}\phi(t)K_n(t)dt + \int_{1/n+1}^{\delta}\phi(t)K_n(t)dt + \int_{\delta}^{\pi}\phi(t)K_n(t)dt$$
$$= I_1 + I_2 + I_3 \text{ (say)} \tag{5.1}$$

In order to prove the theorem, we have to prove that

$$\int_0^{\pi}\phi(t)K_n(t)dt = o(1), \text{ as } n \to \infty$$

Now, we consider

$$|I_1| \le \int_0^{1/n+1} |\phi(t)||K_n(t)|dt$$

$$= O(n)\int_0^{1/n+1} |\phi(t)|dt \text{ (Using Lemma1)}$$

$$= O(n)\left[o\left\{\frac{1}{(n+1)\log(n+1)}\right\}\right] \qquad \text{by (3.2)}$$

$$= o\left\{\frac{1}{\log(n+1)}\right\}$$

$$= o(1) \text{ as } n \to \infty \qquad (5.2)$$

Now,

$$|I_2| \le \int_{1/n+1}^{\delta} |\phi(t)||K_n(t)|dt$$

$$= O\left[\int_{1/n+1}^{\delta} |\phi(t)|\left(\frac{1}{t}\right)dt\right] \text{(using Lemma2)}$$

$$= O\left[\left\{\frac{1}{t}\Phi(t)\right\}_{1/n+1}^{\delta} + \int_{1/n+1}^{\delta} \frac{1}{t^2}\Phi(t)dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha\left(1/t\right)C_\tau}\right\}_{1/n+1}^{\delta} + \int_{1/n+1}^{\delta} o\left\{\frac{1}{t\alpha\left(1/t\right)C_\tau}\right\}dt\right] \qquad \text{by (3.1)}$$

Putting $1/t = u$ in second term

$$= O\left[o\left\{\frac{1}{\alpha(n+1)C_{n+1}}\right\} + \int_{1/\delta}^{n+1} o\left\{\frac{1}{u\alpha(u)C_u}\right\}du\right]$$

$$= o\left\{\frac{1}{\log(n+1)}\right\} + o\left\{\frac{1}{\log(n+1)}\right\}$$

$$= o(1) + o(1) \text{ as } n \to \infty$$

$$= o(1) \text{ as } n \to \infty \qquad (5.3)$$

By Riemann-Lebesgue lemma & by regularity condition of the method of summability

$$|I_3| \le \int_{\delta}^{\pi} |\phi(t)||K_n(t)|dt$$

$$= o(1) \text{, as } n \to \infty \qquad (5.4)$$

Combining $(5.2), (5.3) \& (5.4)$

$$I_1 + I_2 + I_3 = o(1)$$

Hence we proved that

$$t^{(E,1)(N,P_n)} - f(x) = o(1) \text{ as } n \to \infty$$

This completes the proof of theorem1.

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