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CHARACTERIZATION OF VICT GRAPH OF A GRAPH

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ABSTRACT

In this paper we introduce the concept of Vict graph V_n (G) of a graph. Also we determine the number of vertices and edges of V_n (G). Further we characterize the graphs whose Vict graphs are planar, outerplanar, maximal outerplanar and minimally nonouterplanar. Finally we develop a necessary and sufficient condition for the Vict graph whose crossing number is one.

1. INTRODUCATION

We shall restrict ourselves to finite, undirected without isolated vertices, loops or multiple edges. For all definitions and notations see [2] and [4].

The line graph of a graph G is the graph whose vertex set corresponds to the edges of G such that two vertices of L(G) are adjacent if the corresponding edges of G are adjacent. This concept was first studied by Whitney [3] and was studied in [8, 12, 13, 14, 15, 18, 19]. Many other graph valued functions in graph Theory were studied, for example [1, 5, 6, 7, 9, 10, 11, 16, 17].

The following will be useful in the proof of our results.

Theorem A [2]: If G is any planar (p, q) graph with $p \geq 3$, then $q \leq 3p-6$.Furthermore, if G has no triangles then $q \leq 2p-4$.

Theorem B [2]: Every maximal outerplanar graph G with $p \geq 3$ vertices has 2p-3 edges.

Theorem C [7]: A connected graph with $p \geq 2$ vertices is non empty path if and only if $\sum di^2 = 4p-6$.

Theorem D [2]: The graph K_5 and $K_{3,3}$ are nonplanar.

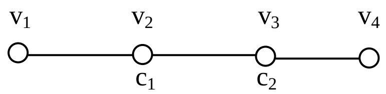
2. VICT GRAPH

We now define the Vict graph V_n (G) of a graph G as the graph whose vertex set is the union of the set of vertices and set of cutvertices of G in which two vertices are adjacent if and only if corresponding vertices of G are adjacent or corresponding members of G are adjacent or incident.

In the Figure1, a graph G and its Vict graph V_n (G) are shown. In V_n (G), the light vertices are corresponding to the vertices of G and the dark vertices corresponding to the cutvertices of G.

A graph G is a subgraph of V_n (G).

G:



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V_n (G):

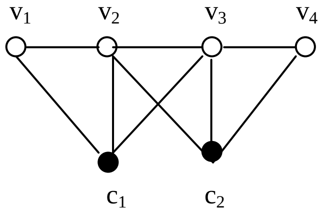

Figure-1

Remark 1: For any non trivial connected graph G, G = V_n (G) if and only if G is a block.

Remark 2: If the degree of a cutvertex in G is n, then the degree of corresponding vertex in V_n (G) is n+1.

Remark 3: If G is a path of length n,(n≥2). Then $|E(V_n (G))|=4n-3$.

Theorem 1: The vict graph V_n (G) of a graph G is connected if and only if G is connected.

Proof: Assume G is disconnected. Then by Remark 1, V_n (G) is disconnected. Hence if G connected, then V_n (G) is connected.

Now assume G is a nonseparable. Then by Remark 1, G= V_n (G) which is connected.

The following theorem determiners the number of vertices and edges in Vict graph of a graph.

Theorem 2: If G is a nontrivial connected (p, q) graph, C_i be the number of cutvertices in G and l_i be the number of edges incident with the cutvertices in G. Then vict graph V_n (G) has p+ $\sum_{i=0}^{n}$ C_i vertices and [$\sum_{i=0}^{n}$ (l_i+C_i)]+q edges.

Proof: Let G be a connected graph with p vertices and q edges. By the definition of vict graph $V_n(G)$ the number of vertices in vict graph $V_n(G)$ is the sum of the vertices of G and cutvertices of G. Thus $V_n(G)$ has P+$\sum_{i=0}^{n}$ Ci vertices. The number of edges in $V_n(G)$ is the sum of the numbers of edges in G also sum of the number of cutvertices in G and number of edges incident to the cutvertices. Thus $V_n(G)$ has $[\sum_{i=0}^{n}(li + Ci)] + q$ edges.

Theorem 3: Let G be a connected (p, q) graph. Then L(G) = n(G) = $V_n(G)$ = G if and only if G is a cycle.

We now present a characterization of graphs whose vict graphs are planar.

For any plane graph G the inner vertex number i(G) of G is the minimum number of vertices not belonging the boundary of the exterior region in any embedding of G in the plane.

We call the inner vertex number i(G) as Kulli number i(G).

A graph G is said to be minimally nonouterplanar if Kulli number is one or i(G) = 1.

Theorem 4: The vict graph $V_n(G)$ of a planar graph G is planar if and only if G satisfies the following conditions.

1) G is a tree.
   Or
2) G does not contain three mutually adjacent cutvertices with Kulli number zero.
   Or
3) G has a block B with Kulli number and no cutvertex of B is adjacent to the Kulli number.
   Or
4) Every block of G is either a cycle or an edge in which at least one vertex of an odd cycle is not a cutvertex.
   Or
5) G has a cycle C_n, n≥4 together with a diagonal edge joining a pair of vertices of any length which are not cutvertices.

Proof: Suppose $V_n(G)$ is planar. Now we consider the following cases.

Case-1: Assume G is not a tree with Kulli number zero. Then there exists a block B in G with Kulli number zero. Suppose B=C_3 and each vertex of B is a cutvertex. Then in $V_n(G)$ the sub graph <$V_n(G)$> forms a subgraph homeomorphic to $K_{3,3}$. Thus $V_n(G)$ is nonplanar, a contradiction.

Case-2: Assume G has a Kulli number and suppose G has a block B with Kulli number and B has cutvertex v which is adjacent to Kulli number of B. Let v_1 v_2, v_3 are adjacent to v and v_4. In $V_n(G)$ the vertex $v^!$ corresponding to the cutvertex v generates an induced subgraph homeomorphic to $K_{3,3}$. Then by the Theorem D, $V_n(G)$ has at least one crossing. Hence $V_n(G)$ is nonplanar, a contradiction.

Case-3: Assume G has a cycle C_n , (n≥4) together with a diagonal edge joining a pair of vertices of length n-2 which are cutvertices. Then in V_n (G), $v^{\prime}_i$ is a cutvertex which is adjacent to v_i, v_{i+1}, v_n , ∀ i=1 and also endvertex which is $N(v^{\prime}_i)$. Similarly a cutvertex $v^{\prime}_{i+2}$ which is adjacent to v_{i+1}, v_{i+2}, v_{i+3}, ∀ i=1 and endvertex which is $N(v^{\prime}_{i+2})$. Where $v^{\prime}_i$ and $v^{\prime}_{i+2}$ are cutvertices corresponding to v_i and v_{i+2}. This adjacency will produce one crossing in V_n (G) in planar embedding of $V_n(G)$ in any plane. Hence $V_n(G)$ is non planar, a contradiction.

Conversely, for (1) suppose G is a tree. Let C= {v_1, v_2, v_3, …,v_n } be the set of cutvertices. In $V_n(G)$, V [$V_n(G)$] = V (G) U {C} and each v_i∈C, 1 ≤ i≤ n adjacent to N(v_i) and v_i . On any embedding of $V_n(G)$ which satisfies $[\sum_{i=0}^{n}(l_i+C_i)]+q \leq 3p-6$. Hence by Theorem A, $V_n(G)$ is planar.

For (2), suppose G has a block B with Kulli number zero. Then G has a set of cutvertices as C= {v_1, v_2, v_3, ,v_n }such that no three cutvertices of a block B are mutually adjacent in G. In $V_n(G)$, V [$V_n(G)$]= V (G) U{C} and each v_i ∈C, 1 ≤ i ≤ n adjacent to N(v_i) and v_i. On any embedding of $V_n(G)$ which satisfies that the number of edges in $V_n(G)$ is less than or equal to 3P-6. Thus by Theorem A, $V_n(G)$ is planar.

For(3), suppose G has a set of blocks as {B_1 ,B_2,…,B_n } such that either each block has i(B_i) ≥1, 1≤i≤m or at least one block of G has i(B_j)≥1, 1≤j≤m. Then no cutvertex of G is adjacent to any of the inner vertex of any B_i, 1≤ i ≤m. In any planar embedding of $V_n(G)$ with ($p^{\prime}$, $q^{\prime}$) satisfies the inequality $q^{\prime} \leq 3P^{\prime}-6$. Hence by Theorem A, $V_n(G)$ is planar.

For (4), suppose G has a odd cycle C_n with vertices v_1, v_2, v_3,…,v_n, v_1 in which at least one vertex of C_n is not a cutvertex. In V_n (G) $v^{\prime}_2$,$v^{\prime}_3$,$v^{\prime}_4$,…..,$v^{\prime}_n$ are cutvertices corresponding to the v_2, v_3, v_4,….,v_n ∈ G . And each cutvertex $v^{\prime}_i$, ∀ i=2,3,4,…,n is adjacent to the v_{i-1}, v_i, v_{i+1} , ∀ i=2,3,4,….,n. And also endvertex which is N ($v^{\prime}_i$). On any embedding of V_n (G) which satisfies that the number of edges in V_n (G) is less than or equal to 3p-6. Thus by Theorem A, V_n (G) is planar.

For (5), suppose G has a cycle C_n (n≥4) with vertices v_1, v_2, v_3,….,v_n, v_1 in which a diagonal edge e joining a pair of vertices of length n-2, which are not cutvertices. But remaining vertices of a cycle C_n are cutvertices. In $V_n(G)$ each cutvertex $v^{\prime}_i$, ∀ i ∈V_n (G) which is adjacent to v_{i-1}, v_i, v_{i+1} , ∀ i ∈ G and also endvertices which are $N(v^{\prime}_i)$ where $v^{\prime}_i$ corresponds to cutvertex v_i ∈ G. On any embedding of V_n (G) which satisfies $[\sum_{i=0}^{n}(\mathrm{li}+\mathrm{Ci})] + q \leq 3p - 6$. Hence by Theorem A, $V_n(G)$ is planar.

In the following Theorem We establish a necessary and sufficient condition for the graphs whose $V_n(G)$ are outerplanar.

Theorem 5: Let G be (p, q) graph. Then vict graph $V_n(G)$ is outerplanar if and only if G is nonseparable outerplanar and G is either a path or a cycle.

Proof: Suppose $V_n(G)$ is outerplanar. Then $V_n(G)$ is connected. Hence G is connected. If $V_n(G)$ is K_2, then obviously G is K_2 . Now assume G is nonseparable, nonouterplanar. Then there exist a Kulli number and by Remark 1, $V_n(G)$=G. Hence $V_n(G)$ is nonouterplanar.

Suppose G is outerplanar and G is neither a path nor a cycle. Then G has at least a vertex v with deg (v) = 3. Now we consider the following cases.

Case-1: Assume v lies on two blocks in which one block is an edge and remaining block is isomorphic to C_3. Then $V_n(G)$ has an induced subgraph$\langle K_4 \rangle$ with Kulli number. Hence $V_n(G)$ is not outerplanar, a contradiction.

Case-2: Assume v lies on three blocks. Then each block incident to v is an edge. Then $V_n(G)$ has a subgraph as $K_{2,3}$, clearly $K_{2,3}$ has a Kulli number. Hence $V_n(G)$ is not outerplanar, a contradiction.

Conversely, suppose G is nonseparable outerplanar and G is either a path or a cycle.

We consider the following cases.

Case-1: Assume G is a block which is outerplanar. Then by Remark 1, $V_n(G)$=G and hence $V_n(G)$ is outerplanar.

Case-2: Assume G is a path with P≥3 vertices. Then by the Theorem B, $V_n(G)$ is maximal outer planar. Hence $V_n(G)$ is outerplanar.

This complete the proof of the theorem.

We now deduce a necessary and sufficient condition for the graphs whose $V_n(G)$ are maximal outerplanar.

Theorem 6: The vict graph $V_n(G)$ of a graph G is maximal outerplanar if and only if G is path.

Proof: Suppose $V_n(G)$ is maximal outerplanar. Then $V_n(G)$ is connected. Hence G is connected. If $V_n(G)$ is K_1 or K_2, then obviously G is K_1 or K_2. Let G be any connected graph with P≥3 vertices with degree d_i and l_i be the number of edges to which the cutvertices C_i belongs in G. Then clearly $V_n(G)$ has $P+\sum C_i$ vertices and $l_i+½\sum d_i^2$ edges.

Since $V_n(G)$ is maximal outerplanar, it has $2(P+\sum C_i)-3$ edges.

Hence $l_i+½\sum d_i^2 =2(P+\sum C_i)-3$ which is the sum as $\sum d_i^2 -4P+6=0$

By Theorem C, it follows that G is a non empty path**.**

Conversely, suppose G is a path. We consider the following cases.

Case-1: Suppose G is K_1 or K_2. Then $V_n(G)$ is K_1 or K_2 and hence it is maximal outerplanar.

Case-2: Suppose G is a non empty path. Now we prove that $V_n(G)$ is maximal outerplanar. This is proved by induction on the number of vertices P(≥2) of G.

It is easy to observe that the vict graph of K_2 is maximal outerplanar by case1.

As the indicative hypothesis, let the vict graph of a non empty path with p=n vertices are maximal outerplanar. We now show that the vict graph of a path $G^!$ with p=n+1 vertices is maximal outerplanar. Let $G^!$ be a path $v_1, v_2, \ldots, v_n, v_{n+1}$ in which $v_2, v_3, \ldots, v_n$ are the cut vertices in $G^!$ and denoted as $v_2^!, v_3^! \ldots, v_n^!$ in $V_n(G^!)$,see Figure2. Consider without loss of generality $G^! - v_{n+1} = G$ is a path with n vertices. By the inductive hypothesis, $V_n(G)$ is maximal outerplanar.

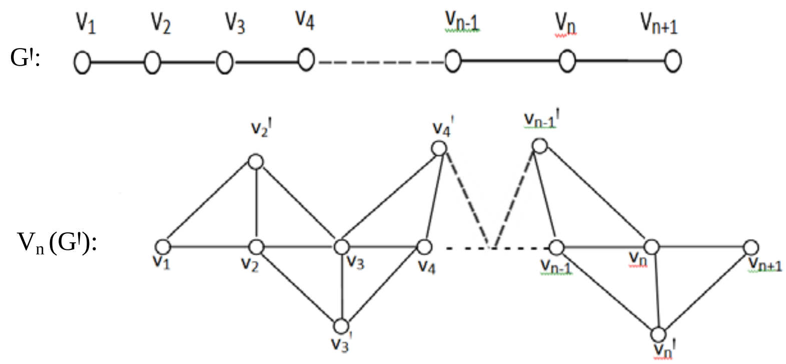

Figure-2

The vertices $v_n^!$ and v_{n+1} are two more vertices in $V_n(G^!)$ than in $V_n(G)$. We also observe that there are only four edges $v_n^! v_{n-1}$, $v_n^! v_n$, $v_n^! v_{n+1}$ and $v_n v_{n+1}$ in $V_n(G^!)$ than in $V_n(G)$. It is clear that the induced subgraph on the vertices v_{n-1}, v_n, v_{n+1}, $v^!_n$ is not K_4. Hence $V_n(G^!)$ is outerplanar. We now prove that $V_n(G^!)$ is maximal outerplanar with 2n-1 vertices and has 2(2n-1)-3 edges. Thus, the outerplanar graph $V_n(G^!)$ has 2n+1 vertices and 2(2n-1)-3+4 =2(2n-1)-3edges. Hence $V_n(G)$ is maximal outerplanar.

We now characterize graphs whose vict graphs are minimally nonouterplanar.

Theorem 7: The vict graph $V_n(G)$ of a graph G is minimally nonouterplanar if and only if G satisfies the following conditions.

1) G is a block with Kulli number one.
   Or
2) G is a path P_n, (n≥3) together with an endedge adjoined to any non endvertex of a path P_n .
   Or
3) G has a triangle together with a path P_n , (n≥2) adjoined to any vertex of a triangle.

Proof: Suppose $V_n(G)$ has a Kulli number. Then $V_n(G)$ is planar.

We consider the following cases:

Case-1: Assume G is a block with Kulli number zero. Then by the Remark 1, $V_n(G)$ =G, a contradiction.

Case-2: Assume G is a block with greater than Kulli number one. Then again by the Remark1, $V_n(G)$=G, a contraction.

In the following case, we consider the separable graph.

Case-3: Assume G is a path P_n, $(n\geq3)$. Then by the Theorem 6, $V_n(G)$ is a maximal outerplanar, a contradiction.

Case-4: *Assume G is not a path. Suppose G is a tree with $\triangle(G) \geq 3$. Then we consider the following subcases.*

Subcase-4.1: *Suppose $\triangle(G)=4$ and G has a cut*vertex v of degree 4. Then graph G contains a subgraph isomorphic to $K_{1,4.}$ Thus $V_n(G)$ has a subgraph as $K_{2,3.}$ Hence $V_n(G)$ contains Kulli number more than one, a contradiction.

Subcase-4.2: Suppose G contains at least two vertices of degree three. Then G contains a subgraph isomorphic to $K_{3,3}$-C_4. Thus $V_n(G)$ has vertices and edge joining $K_{2,3}$ as a subgraph.

Hence $V_n(G)$ has Kulli number greater than one, a contradiction.

Case-5: Assume G is not a block and G is free from Kulli number. Then we consider the following subcases.

Subcase-5.1: Suppose G is free from Kulli number and G has a cycle $C_{3,}$ together with two paths P_n and P_m ,(m, n $\geq$2) adjoined to the adjacent vertices of cycle C_3. Then in $V_n(G)$, $v_{1,}v_{2,}v_3\in C_3$ and $v_{4,}v_5\in N(v_2)UN(v_3)$ and, $v_2^!$, $v_3^!\in V((V_n(G))$ such that $v_2^!$, $v_{1,}$ $v_{2,}$ $v_{3,}$ and v_4 from $K_{2,3}$ as subgraph, similarly v_3, v_1, v_2, $v_{3,}$ and v_5 from another $K_{2,3.}$ Hence in any embedding of $V_n(G)$, it has a Kulli number greater than one, a contradiction.

Subcase-5.2: Suppose G is free from a Kulli number and has cycle $C_{4,}$ together with path a P_n, (n $\geq$2) adjoined to any vertex of cycle $C_{4.}$ Then by the Theorem 4, $V_n(G)$ is planar. On embedding $V_n(G)$ in any plane, one can easily verify that V_n (G) has Kulli number greater than one, a contradiction.

Subcase-5.3: Suppose G has a Kulli number greater than one and a cycle C_n, **(**n $\geq$3), together with an endedge adjoined to any vertex of a cycle C_n .Then in $V_n(G)$ $v_{1,}$ v_2 , $v_{3,\ldots..}$ $v_n\in C_{n\,,}$ $v^!_i \in N(v_i)$ where (i=1, 2,..,n) and $v_i^! \in V(v_n(G))$, where $v_i^!$, corresponding to the cutvertex $v_i \in$ G, and $v_i^!$ adjacent to $N[V(v_i^!)]$ in $V_n(G)$. This adjacency produces either greater than Kulli number one or a nonplanar graph by the Theorem4. On embedding of $V_n(G)$ in any plane, a contradiction.

Conversely, for (1) G has no cutvertex. Thus by Remark1, $V_n(G)$=G. Hence $V_n(G)$ has Kulli number one.

This Proves (1).

For(2), suppose G contains a path P_n (n$\geq$3) with vertices v_i, ɏ i=1, 2, 3….,n. Then an end edge adjoined to any nonend vertex of v_i of a path P_n, I = 2, 3,…..,n-1. And $v^!$ be the endvertex adjacent to $v_{i.}$ In $V_n(G)$, $v_{i,}$ $v_{i-1,}$ $v_{i+-1,}$ $v^!$ and $v_i^!$ forms $K_{2,3}$ as a subgraph in $V_n(G)$. Where $v_i^!$ correspond to the cutvertex $v_i \in P_n$ and remaining regions of $V_n(G)$ are triangulated. Hence $V_n(G)$ has a Kulli number one.

This proves (2).

For (3), suppose G contains a triangle with vertices v_i, (i=1, 2 and 3) and $v_{1,}$ $v_{2,}$ ……, v_n are the vertices of a path P_n,(n $\geq$3). Then any endvertex of a path P_n either v_1 or v_n adjoined to any vertex v_i of a triangle, ɏi $\in$ 1, 2 and 3. In $V_n(G)$ vertices of v_i of a triangle, ɏ i=1, 2, 3 and a vertex v_n of a path P_n, ɏ n $\in$ 1, 2,…, n which is a $N(v_i)$ and $v^!$ forms $K_{2,3}$ as a subgraph in $V_n(G)$.where $v^!$ correspond to cutvertex in $v_i \in$ G, ɏ i $\in$ 1, 2, 3 and remaining regions of $V_n(G)$ are triangulated. Hence $V_n(G)$ has a Kulli number one.

This Proves (3).

Theorem 8: If G has a cycle $C_n(n\geq3)$ together with a path P_n, (n$\geq$2) adjoined to any vertex of a cycle C_n. Then $V_n(G)$ has (n-2) - Kulli number.

Proof: Suppose $V_n(G)$ has greater than or equal to Kulli number one. Then $V_n(G)$ is connected. Hence G is connected. We prove the result by mathematical induction on the number of vertices of a cycle C_n of G.

Suppose n=3. Then G has a cycle C_3 together with a path P_n adjoined to any vertex of a cycle C_3. Thus by Theorem7, $V_n(G)$ has a Kulli number one. Hence the result is true for n=3. Assume that result is true for n=m.

Now we consider a cycle C_m together with a path P_n adjoined to any vertex of a cycle C_m.. Then $V_n(G)$ has (m-2)-Kulli number.

Suppose n=m+1. Then G has a cycle C_{m+1} together with a path P_n adjoined to any vertex of a cycle C_{m+1}. Then we have to prove that $V_n(G)$ has [(m+1)-2]=(m-1) -Kulli number.

Let v_{m+1} be vertex of G and let G= $C_{m+1,}$ delete from G the vertex v_{m+1} by deleting the edges $e_m=(v_{m,}\ v_{m+1})$ and $e_{m+1} = (v_{m+1,}\ v_1)$ which are incident with $v_{m+1,}$ resulting the graph $G_1 = C_m$. By inductive hypothesis $V_n(G)$ has (m-2)-Kulli number.

Now rejoin the vertex v_{m+1} to the vertices v_m and v_1 of G_1 by joining the edges e_m and e_{m+1} which results the graph G. The formation of $V_n(G)$ is an extension of $V_n(G_1)$ with additional vertex v_m and additional edges e_m and $e_{m+1.}$ In $V_n(G)$, without loss of generality, the vertices v_1, v_m and v_{m+1} of a cycle C_{m+1} are adjacent to the $v^!_{m+1}$ Where $v^!_{m+1}$ correspond to a cutvertex $v_{m+1}\in C_{m+1}$. This adjacency in $V_n(G)$ produces (m+1) Kulli number in the interior region of a cycle C_{m+1} and remaining regions of $V_n(G)$ are triangulated. Hence $V_n(G)$ has [(m+1)- 2]=(m-1)-Kulli number.

Theorem 9: No vertex of $V_n(G)$ is a cutvertex.

Proof: Since $V_n(G)$ is a subgraph of G, the only cutvertices of G may be the cutvertices of $V_n(G)$. Thus it is sufficient to show that a cutvertex of a connected graph G is not a cutvertex of $V_n(G)$. Let G contains B_i = (i=1, 2,…,n) as blocks. Then there exists v_i, $1 \le i \le n$ cutvertices in G. In $V_n(G)$, every cutvertex $v_i \in G$ is adjacent to the corresponding $N(v_i)$ and v_i, ∀ i = 1, 2,…,n. This adjacency produced a non separable graph in $V_n(G)$. Hence no vertex of $V_n(G)$ is a cutvertex.

In the following theorem, we develop the result for crossing number of nonplanar graphs.

Theorem 10: The vict graph $V_n(G)$ of K_5 and $K_{3,3}$ has crossing number at least one.

Proof: Suppose G is isomorphic to K_5 or $K_{3,3}$. Then G has no cutvertex. By the Remark1, $V_n(G)$=G. On embedding of $V_n(G)$ in any plane, $V_n(G)$ has at least crossing number one.

In view of the above Theorem we establish the following result for crossing number of a graph which contains a block and is nonplanar.

Theorem 11: The vict graph V_n (G) of $K_{3,3}$ and $K_{5,}$ together with an endedge adjoined to any vertex of $K_{3,3}$ and K_5 has crossing number at least two and three.

Proof: Suppose G has $K_{3,3}$ with vertices $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$. Then vertex set V of $K_{3,3}$ is divided into two subsets as V_1 and V_2 such that each vertex set has three distinct vertices from set V and also no vertex of V_1 set is adjacent to each other. Similarly as vertex set V_2 and $V_1 \cup V_2$ =V. But every vertex of V_1 vertex set is adjacent to all the vertices of V_2 vertex set. An endedge is adjacent to any vertex of V_1 set or V_2 set.

In V_n (G) any one set of vertices either V_1 or V_2 each contains three distinct vertices from V and v_7 of an endedge are adjacent to $v_i^! \in V_n$ (G) where $v_i^!$ correspond to a cutvertex $v_i\in G$, ∀ i=1,2,3,4,5,6. On embedding of V_n (G) in any plane. Produces at least one crossing in $V_n(G)$. By Theorem 10, $K_{3,3}$ has at least one crossing. Hence V_n (G) has at least two crossing.

Suppose G has K_5 with vertices v_i, (i=1, 2, 3, 4, 5) such that each vertices of K_5 are mutually adjacent to each other in G. An endedge is adjoined to any vertex of $v_i \in K_{5.}$ In V_n (G) every vertex $v_i \in K_5$ are adjacent to $v^!$ and also v_6 of an endedge of G is adjacent to $v^!$ where v! is a cut vertex corresponding to $v_i\in G$. This adjacency produces at least two crossings. On any embedding of V_n (G) in any plane. But by Theorem 10, K_5 has at least one crossing. Thus V_n (G) has at least three crossing.

We now present a characterization of planar graph whose vict graph has crossing number 1.

Theorem 12: The vict graph V_n (G) of a graph G has crossing number one if and only if G is planar and (1) or (2) or (3) or (4) holds.

1) G contains three mutually adjacent cutvertices with Kulli number zero.
   Or
2) G contains an odd cycle C_n, $(n\ge3)$ and each vertex of C_n is a cutvertex.

Or

3) G has a block B with Kulli number and a cutvertex of B is adjacent to the Kulli number.

Or

4) G has a cycle C_n, $n \geq 4$ together with a diagonal edge joining a pair of vertices of any length which are cutvertices.

Proof: If G is planar graph satisfying (1) or (2) or (3) or (4) then by Theorem 10, V_n (G) has crossing number at least one. Now we show that its crossing number is at most 1.

First assume (1) holds. Let v_1, v_2, v_3 are three mutually adjacent cutvertices in G. In V_n (G) $v_1^!$, $v_2^!$, $v_3^!$ are the vertices corresponding to the vertices v_1, v_2, v_3 of G. Then v_1, v_2, v_3, $v_1^!$, $v_2^!$ and $v_3^!$ forms a $K_{3,3}$ as an induced subgraph in V_n (G). Hence by Theorem D. C_r [V_n (G)] = 1.

Assume (2) holds. Let G has a cycle C_n, $n \geq 3$, if n=3 then by condition (1), the result is true. For C_n, $n \geq 4$ and n is odd, let C_n: v_1, $v_{2,\ldots\ldots}$,$v_{n,}$ v_1 is a odd cycle and each v_i is a cutvertex $1 \leq i \leq n$, consider x and y are the vertices adjacent to v_1 and v_n [see Fig 3(a)]. On embedding V_n (G) in any plane the vertices $v_1^!$ corresponds to v_1 is adjacent to v_2, v_1, v_n and x, similarly $v_n{}^!$ corresponds to v_n is adjacent to v_1, $v_{n,}$ v_{n-1} and Y. thus the edges $v_1{}^! v_n$ and $v_1 v_n{}^!$ are intersecting with one crossing [see Fig 3(b)]. The remaining cut vertices{$v_2^!$,$v_3^!$,..,$V_{n-1}^!$} corresponding to{v_2, v_3,…,v_{n-1}}are adjacent to $N(v_j)$ $2 \leq j \leq n-1$ without any crossing. Hence V_n (G) has crossing number one.

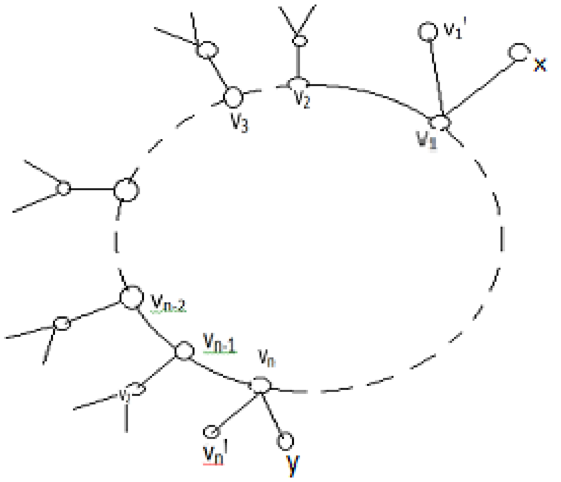

Figure-[3(a)]

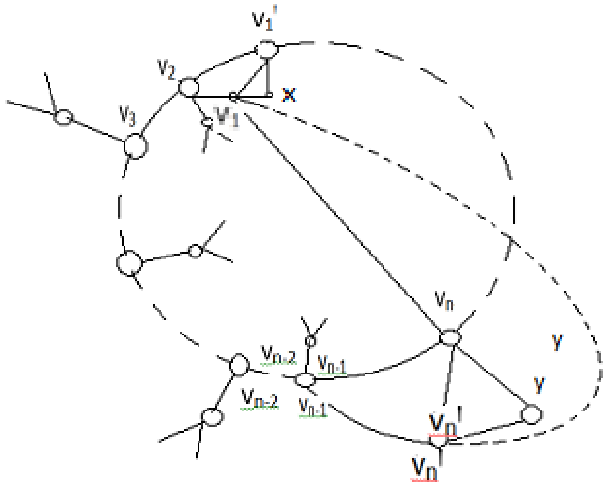

Figure-[3(b)]

Further (3) holds. For this condition we consider the smallest Kulli number in G which generates a graph contains exactly two blocks, one is $K_{2,3}$ and other as an edge e. Let v be the cutvertex adjacent to Kulli number. In depicting V_n (G), it has a subgraph homomorphic to $K_{3,3.}$ Hence V_n (G) has crossing number one.

Assume (4) holds. Let G has a cycle $C_{n,}$ $n\geq4$, let $v_1,v_2,v_3,\ldots,v_n,v_1$ is a cycle and a diagonal edge e joining pair of vertices v_i and v_{i+3}, ɏ i=1, consider x and y are vertices adjacent to v_i and v_{i+3} [see Fig 4(a)]. On embedding $V_n(G)$ in any plane the vertex v'_i corresponds to v_i, ɏ i=1 is adjacent to v_i, v_{i+1}, v_{i+3}, v_n and x, ɏ i=1, similarly v'_{i+3} corresponds to v_{i+3} adjacent to v_{i+2}, v_{i+3}, v_{i+4}, v_i and y, ɏ i=1. Thus the edges $v_i\, v_{i+3}$ and $v_{i+1}\, v'_i$, ɏ i=1 are intersecting with one crossing [see Fig 4(b)]. Hence V_n (G) has crossing number one.

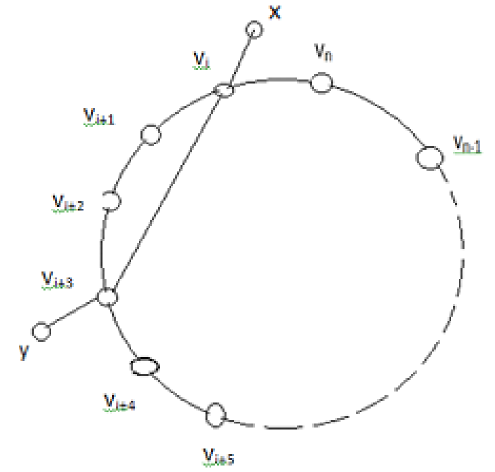

Figure-[4(a)]

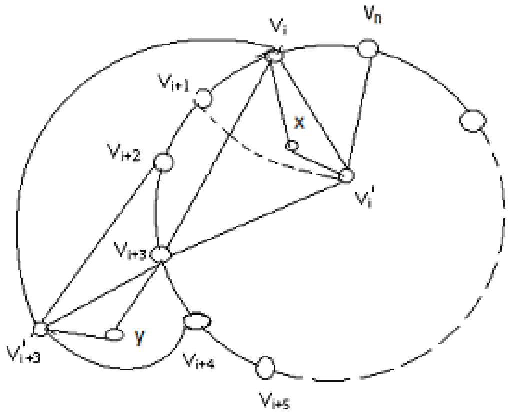

Figure-[4(b)]

For the converse, suppose V_n (G) has crossing number one. By Theorem10, G is planar.

We consider the following cases.

Case-1: Assume G has a block B with Kulli number zero.

Again we have the following sub cases.

Subcase-1.1: Suppose C= {v_1, v_2 …, v_k}be the set of cutvertices. If no any three vertices of C are mutually adjacent. Then by Theorem 4, V_n (G) has crossing number zero, a contradiction.

Subcase-1.2: Suppose there exists two sets A={v_i, v_j, v_k} i=1, j=1, k=1 and B={v_l, v_m, v_n} with l=1, m=1, n=1, such that every element of A and B are mutually adjacent and $A,B\in C$. Then in V_n (G) there two sets gives two subgraph which are isomorphic to $K_{3,3}$. Hence $C_r\,[V_n\,(G)] >1$, a contradiction.

Subclass-1.3: suppose G has only even cycle and each vertex of C_n is a cutvertex. Then by Theorem 4. $C_r\,[V_n\,(G)] = 0$, a contraction.

Subcase-1.4: Suppose G has at least one odd cycle C_n and at least one vertex $v\in C_n$ is not a cutvertex. Then by Theorem 4. $C_r\,[V_n(G)] = 0$, a contraction.

Subcase-1.5: Suppose G has only a cycle C_n ($n\geq4$) together with a diagonal edge joining pair vertices of any length which are not cutvertices. Then by Theorem 4, $C_r\,[V_n(G)]=0$, a contraction.

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