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SOME PROPERTIES OF LINEAR COMBINATIONS OF TWO IDEMPOTENT MATRICES OVER SKEW FIELD

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ABSTRACT

In this paper, idempotent matrices over skew field have been researched and some properties of idempotent matrices were extended from general complex domain to skew field. In this paper, the following conclusions were obtained: (1) the four equivalent conditions of idempotent matrices over skew field;(2) the necessary and sufficient conditions which could infer that the linear combinations A_1+A_2 and A_1-A_2 of idempotent matrices over skew field A_1, A_2 were also idempotent matrices; (3) the necessary and sufficient conditions of nonsingularity of correlative left linear combinations $c_1A_1+c_2A_2$ and $c_1A_1A_2+c_2A_2A_1$, where $c_1,c_2\in K_{A_1,A_2}$ of idempotent matrices over skew field A_1,A_2.

Keywords: *Idempotent Matrices over Skew Field; Linear Combinations; Idempotency; Nonsingularity.*

1. INTRODUCTION

Some properties of linear combinations of idempotent matrices in general complex domain had been proved in the papers available. In 2011, Liu xiaochuan and He mei studied idempotent matrices in number field F using the theory and method of linear space and obtained some equivalent conditions of idempotent matrices [3]; then the equivalent conditions of idempotent matrices in number field were extended to skew field in this paper and four equivalent conditions of idempotent matrices over skew field were achieved. In addition, the relationship between idempotent matrices over skew field and their right null space, right column space and rank were discussed, meanwhile, the necessary and sufficient conditions which could infer that the linear combinations A_1+A_2 and A_1-A_2 of idempotent matrices over skew field A_1,A_2 were also idempotent matrices were proved. In 2006, Shan jun researched the nonsingularity problems of non trivial linear combinations of two idempotent matrices by using the null space of the matrices in complex domain and carried out several necessary and sufficient conditions of nonsingularity of linear combinations of two idempotent matrices[6];however, in this paper, those conclusions were extended to skew field and the necessary and sufficient conditions of nonsingularity of correlative left linear combinations $c_1A_1+c_2A_2$ and $c_1A_1A_2+c_2A_2A_1$ $\left(\text{where } c_1,c_2\in K_{A_1,A_2}\right)$ of idempotent matrices over skew field A_1, A_2 were summarized.

In this paper, let K be a skew field, $K^{m\times n}$ represents the set of the unit $m\times n$ matrix, $M_n(K)$ represents the set of the unit $n\times n$ matrix , I_n is $n\times n$ identity matrix over skew field and $K^n=K^{n\times 1}$. $R_r(A)=\left\{AX\middle|X\in K^n\right\}$ and $N_r(A)=\left\{X\in K^n\middle|\ AX=0\right\}$ means the subspace of right vector space K^n which are called right column space and right null space of A respectively. $K_{A_1,A_2}=\left\{x\middle|x \text{ is commutative for all of the elements of } A_1 \text{ and } A_2\right\}$.

Lemma 1: $K^n=R_r(A)+R_r(I_n-A)$.

Lemma 2: $\dim R_r(A)=r(A),\dim N_r(A)=n-r(A)$.

Lemma 3: Let $A\in M_n(K)$, then A is nonsingular if and only if $N_r(A)=\{0\}$.

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2. MAIN CONCLUSIONS

(1)The idempotency of linear combinations of two idempotent matrices over skew field

Theorem 1: Let $A \in M_n(K)$, then the following statements are equivalent:
(1) $A^2 = A$; (2) $N_r(A) = R_r(I_n - A)$; (3) $r(A) + r(I_n - A) = n$; (4) $K^n = R_r(A) \oplus R_r(I_n - A)$.

Proof:
$(1) \Rightarrow (2)$: For arbitrary $x \in N_r(A)$, $Ax = 0$ is known, so $x = x - Ax = (I_n - A)x \in R_r(I_n - A)$. Conversely, for every $x \in R_r(I_n - A)$, there exists $y \in K^n$, such that $x = (I_n - A)y$, then $Ax = (A - A^2)y = 0$, and that is $x \in N_r(A)$, hence $N_r(A) = R_r(I_n - A)$.

$(2) \Rightarrow (3)$: From Lemma 2, $r(I_n - A) = \dim R_r(I_n - A) = \dim N_r(A) = n - r(A)$, thus $r(A) + r(I_n - A) = n$.

$(3) \Rightarrow (4)$: According to Lemma 1,it is clear that $K^n = R_r(A) + R_r(I_n - A)$ and $r(A) + r(I_n - A) = n$, so

$$\begin{aligned}
\dim K^n = n &= \dim\left(R_r(A) + R_r(I_n - A)\right) \\
&= \dim R_r(A) + \dim(I_n - A) - \dim\left(R_r(A) \cap R_r(I_n - A)\right) \\
&= r(A) + r(I_n - A) - \dim\left(R_r(A) \cap R_r(I_n - A)\right) \\
&= n - \dim\left(R_r(A) \cap R_r(I_n - A)\right)
\end{aligned}$$

It is obvious that $\dim\left(R_r(A) \cap R_r(I_n - A)\right) = 0$, and that is $R_r(A) \cap R_r(I_n - A) = \{0\}$, accordingly,

$$K^n = R_r(A) \oplus R_r(I_n - A).$$

$(4) \Rightarrow (1)$: For arbitrary $x \in K^n$, $A(I_n - A)x \in R_r(A)$, $(I_n - A)Ax \in R_r(I_n - A)$, because $A(I_n - A)x = (I_n - A)Ax = (A - A^2)x$, $(A - A^2)x \in R_r(A) \cap R_r(I_n - A) = \{0\}$, from which $(A - A^2)x = 0$ is known, which illustrates $A = A^2$.

Theorem 2: Let A_1, $A_2 \in M_n(K)$.If A_1, A_2 are all idempotent matrices, then $A_1 + A_2$ is idempotent matrix if and only if $A_1A_2 = A_2A_1 = 0$ which follows that

$$R_r(A_1) \oplus R_r(A_2) = R_r(A_1 + A_2) \text{ and } N_r(A_1) \cap N_r(A_2) = N_r(A_1 + A_2).$$

Proof: Sufficiency: In view of the conditions above, $(A_1 + A_2)^2 = (A_1 + A_2)(A_1 + A_2)$
$= A_1^2 + A_1A_2 + A_2A_1 + A_2^2 = A_1 + A_2$, which shows that $A_1 + A_2$ is idempotent matrix.

Necessity: Because of what is known, $(A_1 + A_2)^2 = A_1 + A_2 = A_1^2 + A_1A_2 + A_2A_1 + A_2^2 = A_1 + A_2 + A_1A_2 + A_2A_1$, which leads to $A_1A_2 + A_2A_1 = 0$ and that is $A_1A_2 = -A_2A_1$ .Further, $A_1A_2 = -A_1A_2A_1 = A_2A_1A_1 = A_2A_1$,so $A_1A_2 = A_2A_1 = 0$,from which, it is clear that $R_r(A_1) \subseteq N_r(A_2)$, $R_r(A_2) \subseteq N_r(A_1)$, and $R_r(A_1) \cap R_r(A_2) = \{0\}$
.
Obviously, $R_r(A_1 + A_2) \subseteq R_r(A_1) \oplus R_r(A_2)$.For any $x \in R_r(A_1)$, if there exists $y \in K^n$ such that

$$x = A_1y = A_1^2y = (A_1^2 + A_2A_1)y = (A_1 + A_2)A_1y, \text{ then } R_r(A_1) \subseteq R_r(A_1 + A_2).$$

Similarly, $R_r(A_2) \subseteq R_r(A_1 + A_2)$ and then $R_r(A_1) \oplus R_r(A_2) \subseteq R_r(A_1 + A_2)$, consequently,

$$R_r(A_1) \oplus R_r(A_2) = R_r(A_1 + A_2).$$

Besides, it is easy to know that $N_r(A_1) \cap N_r(A_2) \subseteq N_r(A_1 + A_2)$. Moreover, for every $x \in N_r(A_1 + A_2)$, $A_1x = A_1^2x = (A_1^2 + A_1A_2)x = A_1(A_1 + A_2)x = 0 = A_2(A_1 + A_2)x = (A_2A_1 + A_2^2)x = A_2^2x = A_2x$, then $N_r(A_1 + A_2) \subseteq N_r(A_1) \cap N_r(A_2)$. Above with the previous, $N_r(A_1) \cap N_r(A_2) = N_r(A_1 + A_2)$.

Corollary 1: Let $A_1, A_2 \in M_n(K)$. If A_1, A_2 are all idempotent matrices, then $A_1 - A_2$ is idempotent matrix if and only if $A_1A_2 = A_2A_1 = A_2$ which follows that

$$N_r(A_1) \oplus R_r(A_2) = N_r(A_1 - A_2) \text{ and } R_r(A_1) \cap N_r(A_2) = R_r(A_1 - A_2).$$

Proof:

Sufficiency: In terms of what are given in the theorem,
$(A_1 - A_2)^2 = A_1^2 - A_1A_2 - A_2A_1 + A_2^2 = A_1 + A_2 - A_1A_2 - A_2A_1 = A_1 - A_2$, which means that $A_1 - A_2$ is idempotent matrix.

Necessity: From the conditions above, $I_n - (A_1 - A_2)$ is idempotent matrix and so $(I_n - A_1) + A_2$. In the same way, $I_n - A_1$ is idempotent matrix, too. On the basis of Theorem 2, $(I_n - A_1)A_2 = A_2(I_n - A_1) = 0$, which completes $A_1A_2 = A_2A_1 = A_2$. Further,

$$R_r(I_n - A_1) \oplus R_r(A_2) = R_r(I_n - A_1 + A_2) = R_r(I_n - (A_1 - A_2)) \text{ and}$$
$$N_r(I_n - A_1) \cap N_r(A_2) = N_r(I_n - A_1 + A_2) = N_r(I_n - (A_1 - A_2)).$$

Combined with $(1) \Rightarrow (2)$ in Theorem 1,

$$N_r(A_1) \oplus R_r(A_2) = N_r(A_1 - A_2) \text{ and } R_r(A_1) \cap N_r(A_2) = R_r(A_1 - A_2).$$

(2) The nonsingularity of left linear combinations of two idempotent matrices over skew field

Theorem 3: Let A_1, $A_2 \in M_n(K)$ and A_1, A_2 be idempotent matrices. If a left linear combination $\tilde{c}_1A_1 + \tilde{c}_2A_2$ about A_1 and A_2 is nonsingular for some nonzero $\tilde{c}_1, \tilde{c}_2 \in K_{A_1,A_2}$ satisfying $\tilde{c}_1 + \tilde{c}_2 \neq 0$, then $c_1A_1 + c_2A_2$ is nonsingular for all nonzero $c_1, c_2 \in K_{A_1,A_2}$ satisfying $c_1 + c_2 \neq 0$.

Proof: For every nonzero $\tilde{c}_1, \tilde{c}_2 \in K_{A_1,A_2}$ such that $\tilde{c}_1 + \tilde{c}_2 \neq 0$, consider $x \in N_r(c_1A_1 + c_2A_2)$, then

$$(c_1A_1 + c_2A_2)x = 0 \text{ and so } c_1A_1x = -c_2A_2x \tag{2.1}$$

Premultiplying both sides of (2-1) by A_1, A_2 respectively yields

$$c_1A_1x = -c_2A_1A_2x \tag{2.2}$$
$$c_1A_2A_1x = -c_2A_2x \tag{2.3}$$

From (2.1), (2.2), (2.3) and notice that $c_1 \neq 0, c_2 \neq 0$, then

$$A_2x = A_1A_2x,\quad A_1x = A_2A_1x \tag{2.4}$$

However, $(\tilde{c}_1A_1 + \tilde{c}_2A_2)^2 = \tilde{c}_1^2A_1 + \tilde{c}_1\tilde{c}_2A_1A_2 + \tilde{c}_1\tilde{c}_2A_2A_1 + \tilde{c}_2^2A_2$, according to (2.4), then

$$\begin{aligned}(\tilde{c}_1A_1 + \tilde{c}_2A_2)^2 x &= \tilde{c}_1^2A_1x + \tilde{c}_1\tilde{c}_2A_1A_2x + \tilde{c}_1\tilde{c}_2A_2A_1x + \tilde{c}_2^2A_2x \\ &= \tilde{c}_1^2A_1x + \tilde{c}_1\tilde{c}_2A_2x + \tilde{c}_1\tilde{c}_2A_1x + \tilde{c}_2^2A_2x \\ &= \tilde{c}_1(\tilde{c}_1 + \tilde{c}_2)A_1x + \tilde{c}_2(\tilde{c}_1 + \tilde{c}_2)A_2x \\ &= (\tilde{c}_1 + \tilde{c}_2)(\tilde{c}_1A_1 + \tilde{c}_2A_2)x\end{aligned}$$

Under the conditions that $\tilde{c}_1A_1 + \tilde{c}_2A_2$ is nonsingular, then

$$(\tilde{c}_1 + \tilde{c}_2)x = (\tilde{c}_1A_1 + \tilde{c}_2A_2)x = \tilde{c}_1A_1x + \tilde{c}_2A_2x \tag{2.5}$$

Premultiplying both sides of (2-5) by A_1 entails $\tilde{c}_1A_1x + \tilde{c}_2A_1x = \tilde{c}_1A_1x + \tilde{c}_2A_1A_2x$ and that is $A_1x = A_1A_2x$. In the light of (2-2), $(c_1 + c_2)A_1x = 0$ and then $A_1x = 0 = A_1A_2x$.

Combining those with (2-4), it is sure that $A_2x=0$ .Evidently, $(\tilde{c}_1+\tilde{c}_2)x=0$ according to (2-5), which under the assumption that $\tilde{c}_1+\tilde{c}_2\neq 0$ is equivalent to $x=0$ .This means that $N_r(c_1A_1+c_2A_2)=\{0\}$. From Lemma 3, $c_1A_1+c_2A_2$ is nonsingular.

Corollary 2: Let $A_1,A_2\in M_n(K)$, and A_1, A_2 be idempotent matrices. If A_1+A_2 is nonsingular, then for all nonzero $c_1,c_2\in K_{A_1,A_2}$ satisfying $c_1+c_2\neq 0$, $c_1A_1+c_2A_2$ is nonsingular, too.

Theorem 4: Let A_1, $A_2\in M_n(K)$, and A_1, A_2 be idempotent matrices, then for any nonzero $c_1,c_2\in K_{A_1,A_2}$,the following statements are equivalent:（1） A_1-A_2 is nonsingualr; (2) $c_1A_1+c_2A_2$ and $I_n-A_1A_2$ are nonsingular.

Proof:

$(1)\Rightarrow(2)$: From the proof of Theorem 3,it is known that if $x\in N_r(c_1A_1+c_2A_2)$, then x satisfies equalities (2.4), which implicates that $(A_1-A_2)^2x=(A_1^2-A_1A_2-A_2A_1+A_2^2)x=A_1^2x-A_1A_2x-A_2A_1x+A_2^2x=0$. Moreover, A_1-A_2 is nonsingualr, then $x=0$, which means that $N_r(c_1A_1+c_2A_2)=\{0\}$, then $c_1A_1+c_2A_2$ is nonsingular. In a similar way, for any $x\in N_r(I_n-A_1A_2)$, $(I_n-A_1A_2)x=0$ and that is $x=A_1A_2x$ .Premultiplying both sides of the equality above by A_1, A_2, respectively entails $A_1x=A_1A_2x=x$ and $A_2A_1x=A_2x$, so

$$(A_1-A_2)^2x=A_1x-A_1A_2x-A_2A_1x+A_2x=0.$$

As previously, $N_r(I_n-A_1A_2)=\{0\}$, and then $I_n-A_1A_2$ is nonsingular.

$(2)\Rightarrow(1)$: For every $x\in N_r(A_1-A_2)$, $(A_1-A_2)x=0$ and then $A_1x=A_2x$.
Premultiplying both sides of the equality above by A_1, A_2, respectively yields $A_1x=A_1A_2x$ and $A_2x=A_2A_1x$, so $(c_1A_1+c_2A_2)(I_n-A_1A_2)x=(c_1A_1+c_2A_2-c_1A_1A_2-c_2A_2A_1A_2)x=c_2A_2x-c_2A_2A_1x=0$ Furthermore, $c_1A_1+c_2A_2$ and $I_n-A_1A_2$ are all nonsingular, then $x=0$,which implicates that $N_r(A_1-A_2)=\{0\}$,thus A_1-A_2 is nonsingualr.s

Corollary 3: Let A_1, $A_2\in M_n(K)$, and A_1, A_2 be idempotent matrices, then the following statements are equivalent:（1） A_1-A_2 is nonsingualr; (2) A_1+A_2 and $I_n-A_1A_2$ are nonsingular.

Theorem 5: For any $A_1,A_2\in M_n(K)$ which satisfies A_1, A_2 are idempotent matrices, and any nonzero $c_1,c_2\in K_{A_1,A_2}$ such that $c_1+c_2\neq 0$, then $c_1A_1+c_2A_2$ is nonsingular if and only if

$$R_r(A_1(I_n-A_2))\cap R_r(A_2(I_n-A_1))=\{0\} \text{ as well as } N_r(A_1)\cap N_r(A_2)=\{0\}.$$

Proof: Sufficiency: For arbitrary $x\in N_r(c_1A_1+c_2A_2)$, from (2.2)、(2.3)、(2.4) in the proof of Theorem 3,it is known that $(c_1+c_2)A_1x=c_1A_1x+c_2A_1x=-c_2A_1A_2x+c_2A_1x=c_2A_1(I_n-A_2)x$, then $A_1x\in R_r(A_1(I_n-A_2))$. In addition, $(c_1+c_2)A_1x=-c_2A_2x+c_2A_2A_1x=-c_2A_2(I_n-A_1)x$,so $A_1x\in R_r(A_2(I_n-A_1))$ and therefore $A_1x\in R_r(A_1(I_n-A_2))\cap R_r(A_2(I_n-A_1))=\{0\}$, which impletes $A_1x=0$ .Similarly, $A_2x=0$ .In terms of all of the above, $x\in N_r(A_1)\cap N_r(A_2)=\{0\}$,then $x=0$ .Obviously, $N_r(c_1A_1+c_2A_2)=\{0\}$, so $c_1A_1+c_2A_2$ is nonsingular.

Necessity: From what is known in the conditions, for any $x\in R_r(A_1(I_n-A_2))\cap R_r(A_2(I_n-A_1))$, if there exists α, $\beta\in K^n$ such that

$$x=A_1(I_n-A_2)\alpha=A_1^{\,2}((I_n-A_2)\alpha)\in R_r(A_1)$$

and $$x=A_2(I_n-A_1)\beta=A_2^{\,2}((I_n-A_1)\beta)\in R_r(A_2),$$ then $x=A_1x=A_2x$.

Hence,

$$\begin{aligned} c_1(c_1A_1+c_2A_2)x &= c_1(c_1A_1x+c_2A_2x) \\ &= c_1(c_1A_1x+c_2A_1x) \\ &= c_1(c_1+c_2)A_1x = c_1(c_1+c_2)x \\ &= (c_1+c_2)\left[c_1x+\left(c_2A_2-c_2A_2^2\right)\alpha\right] \\ &= (c_1+c_2)\left[c_1x+c_2A_2(I_n-A_2)\alpha\right] \\ &= (c_1+c_2)\left[c_1A_1(I_n-A_2)\alpha+c_2A_2(I_n-A_2)\alpha\right] \\ &= (c_1+c_2)(c_1A_1+c_2A_2)(I_n-A_2)\alpha \end{aligned}$$

Besides, $c_1A_1+c_2A_2$ is nonsingular, then $c_1x=(c_1+c_2)(I_n-A_2)\alpha$. Premultiplying both sides of the above equality by A_1 produces $c_1A_1x=(c_1+c_2)A_1(I_n-A_2)\alpha=(c_1+c_2)x$, and that is $c_1x=(c_1+c_2)x$,then $c_2x=0$. It is easy to know that $x=0$, consequently, $R_r(A_1(I_n-A_2))\cap R_r(A_2(I_n-A_1))=\{0\}$.

On the other hand, for every $x\in N_r(A_1)\cap N_r(A_2)$, $A_1x=0, A_2x=0$,and then $(c_1A_1+c_2A_2)x=c_1A_1x+c_2A_2x=0.$

Since $c_1A_1+c_2A_2$ is nonsingular, $x=0$, and then $N_r(A_1)\cap N_r(A_2)=\{0\}$.

Theorem 6: Let A_1, $A_2\in M_n(K)$, and A_1, A_2 be idempotent matrices. If there exists nonzero $c_1,c_2\in K_{A_1,A_2}$ satisfying $c_1+c_2\neq 0$, then $c_1A_1A_2+c_2A_2A_1$ is nonsingular if and only if $c_1A_1+c_2A_2$ and $I_n-A_1-A_2$ are all nonsingular.

Proof: For $(c_1A_1+c_2A_2)(I_n-A_1-A_2)=c_1A_1+c_2A_2-c_1A_1^2-c_2A_2A_1-c_1A_1A_2-c_2A_2^2=-(c_1A_1A_2+c_2A_2A_1)$, then $c_1A_1A_2+c_2A_2A_1$ is nonsingular if and only if $c_1A_1+c_2A_2$ and $I_n-A_1-A_2$ are all nonsingular.

Corollary 4: Let A_1, $A_2\in M_n(K)$, and A_1, A_2 be idempotent matrices, then $A_1A_2+A_2A_1$ is nonsingular if and only if A_1+A_2 and $I_n-A_1-A_2$ are all nonsingular.

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