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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN G-METRIC SPACES

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ABSTRACT

In this paper we prove common fixed theorem using the concept of weakly compatible mappings. The results can be considered as an extension of theorems to G-metric space.

Key Words: *Weakly compatible mapping, Complete G-metric space.*

INTRODUCTION

The study of common fixed points of mappings satisfying various contractive conditions has been the centre of rigorous research activity. The concept of commuting maps was replaced by weakly commuting maps introduced by Sessa [4]. The concept of weakly commuting maps was further generalized by Jungck who introduced the concept of compatible mappings [1, 2, 3] and then weakly compatible mappings. Common fixed point theorems have been proved using these concepts in metric space, 2 metric space. Sims [5, 6] have shown that most of the results related to D-metric spaces are invalid. Therefore they introduced the concept of. G-metric space. The purpose of this paper is to obtain common fixed point theorem for mappings satisfying condition of weak compatibility

Theorem: Let (X, G) be a complete G-metric Space. Let f and g be self mappings of X satisfying the following conditions

(i) $f(X) \subseteq g(X)$, f(X) is closed subset of g(X)
(ii) f and g are compatible mappings
(iii) $G(fx, fy, fz) \leq \alpha\, G(gx, gy, gz) + \beta\, G(gx, fx, fx) + \gamma\, G(gy, fy, fy) + \delta\, G(gz, fz, fz) + \eta \max\{G(gx, gy, gz), G(gx, fx, fx), G(gx, fy, fy), G(gy, fy, fy), G(gy, fx, fx), G(gy, fz, fz), G(gz, fz, fz), G(gz, fx, fx), G(gz, fy, fy)\}$ for all $x, y, z, \in X$, where $\alpha + \beta + \gamma + \delta + \eta \leq 1/2$

Then f and g have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. By (1) we can choose a point x_1 in X such that $fx_0 = gx_1$, In general we can choose x_{n+1} such that

$$y_n = fx_n = gx_{n+1},\ n = 0, 1, 2, \ldots.$$

From (3) we can write

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+2}) \leq{} & \alpha\, G(gx_n, gx_{n+1}, gx_{n+2}) + \beta\, G(gx_n, fx_n, fx_n) + \gamma\, G(gx_{n+1}, fx_{n+1}, fx_{n+1}) + \delta\, G(gx_{n+2}, fx_{n+2}, fx_{n+2}) \\ & + \eta \max\{G(gx_n, gx_{n+1}, gx_{n+2}), G(gx_n, fx_n, fx_n), G(gx_n, fx_{n+1}, fx_{n+1}), G(gx_{n+1}, fx_{n+1}, fx_{n+1}), \\ & G(gx_{n+1}, fx_n, fx_n), G(gx_{n+1}, fx_{n+2}, fx_{n+2}), G(gx_{n+2}, fx_{n+2}, fx_{n+2}), G(gx_{n+2}, fx_n, fx_n), \\ & G(gx_{n+2}, fx_{n+1}, fx_{n+1})\} \end{aligned}$$

Or

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) \leq{} & \alpha\, G(y_{n-1}, y_n, y_{n+1}) + \beta\, G(y_{n-1}, y_n, y_n) + \gamma\, G(y_n, y_{n+1}, y_{n+1}) + \delta\, G(y_{n+1}, y_{n+2}, y_{n+2}) \\ & + \eta \max\{G(y_{n-1}, y_n, y_{n+1}), G(y_{n-1}, y_n, y_n), G(y_{n-1}, y_{n+1}, y_{n+1}), G(y_n, y_{n+1}, y_{n+1}), G(y_n, y_n, y_n), \\ & G(y_n, y_{n+1}, y_{n+1}), G(y_{n+1}, y_{n+2}, y_{n+2}), G(y_{n+1}, y_n, y_n), G(y_{n+1}, y_{n+1}, y_{n+1})\} \end{aligned}$$

$$\begin{aligned} \leq{} & \alpha\, G(y_{n-1}, y_n, y_{n+1}) + 2\,\beta\, G(y_{n-1}, y_n, y_{n+1}) + 2\,\gamma\, G(y_n, y_{n+1}, y_{n+2}) + 2\,\delta\, G(y_n, y_{n+1}, y_{n+2}) \\ & + \eta \max\{G(y_{n-1}, y_n, y_{n+1}), 2G(y_{n-1}, y_n, y_{n+1}), 2G(y_{n-1}, y_n, y_{n+1}), 2G(y_n, y_{n+1}, y_{n+2}), 2G(y_n, y_{n+1}, y_{n+2}), \\ & 2\, G(y_n, y_{n+1}, y_{n+2}), 2\, G(y_n, y_{n+1}, y_{n+2})\} \end{aligned}$$

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$\leq \alpha\, G(y_{n-1}, y_n, y_{n+1}) + 2\beta\, G(y_{n-1}, y_n, y_{n+1}) + 2\gamma\, G(y_n, y_{n+1}, y_{n+2}) + 2\delta\, G(y_n, y_{n+1}, y_{n+2})$
$+ \eta \max \{2\, G(y_{n-1}, y_n, y_{n+1}),\ 2\, G(y_n, y_{n+1}, y_{n+2})\}$

$\leq (\alpha + 2\beta)\, G(y_{n-1}, y_n, y_{n+1}) + (2\gamma + 2\delta)\, G(y_n, y_{n+1}, y_{n+2}) + \eta \max \{2\, G(y_{n-1}, y_n, y_{n+1}),$
$2\, G(y_n, y_{n+1}, y_{n+1})\}$

We have following cases

Case -I: max = $2\, G(y_{n-1}, y_n, y_{n+1})$

$G(y_n, y_{n+1}, y_{n+2}) \leq (2\eta + \alpha + 2\beta)\, G(y_{n-1}, y_n, y_{n+1}) + (2\gamma + 2\delta)\, G(y_n, y_{n+1}, y_{n+2})$

$$G(y_n, y_{n+1}, y_{n+2}) \leq \left[\frac{2\eta+\alpha+2\beta}{1-2\gamma-2\delta}\right] G(y_{n-1}, y_n, y_{n+1})$$

$G(y_n, y_{n+1}, y_{n+2}) \leq G(y_{n-1}, y_n, y_{n+1})$ as $\alpha + \beta + \delta + \gamma + \eta \leq 1/2$

Case- II: max = $2\, G(y_n, y_{n+1}, y_{n+2})$

$G(y_n, y_{n+1}, y_{n+2}) \leq (\alpha + 2\beta)\, G(y_{n-1}, y_n, y_{n+1}) + (2\gamma + 2\delta + 2\eta)\, G(y_n, y_{n+1}, y_{n+2})$

$$G(y_n, y_{n+1}, y_{n+2}) \leq \left[\frac{\alpha+2\beta}{1-2\gamma-2\delta-2\eta}\right] G(y_{n-1}, y_n, y_{n+1})$$

$G(y_n, y_{n+1}, y_{n+2}) \leq G(y_{n-1}, y_n, y_{n+1})$ as $\alpha + \beta + \delta + \gamma + \eta \leq 1/2$

Similarly now we find

$G(fx_{n+1}, fx_{n+2}, fx_{n+3}) \leq \alpha\, G(gx_{n+1}, gx_{n+2}, gx_{n+3}) + \beta\, G(gx_{n+1}, fx_{n+1}, fx_{n+1}) + \gamma\, G(gx_{n+2}, fx_{n+2}, fx_{n+2})$
$+ \delta\, G(gx_{n+3}, fx_{n+3}, fx_{n+3}) + \eta \max\{G(gx_{n+1}, g_{n+2}, g_{n+3}), G(gx_{n+1}, fx_{n+1}, fx_{n+1}),$
$G(gx_{n+1}, fx_{n+2}, fx_{n+2}),\ G(gx_{n+2}, fx_{n+2}, fx_{n+2}), G(gx_{n+2}, fx_{n+1}, fx_{n+1}),\ G(gx_{n+2}, fx_{n+3}, fx_{n+3}),$
$G(gx_{n+3}, fx_{n+3}, fx_{n+3}),\ G(gx_{n+3}, fx_{n+1}, fx_{n+1}), G(gx_{n+3}, fx_{n+2}, fx_{n+2})\}$

Or

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq \alpha\, G(y_n, y_{n+1}, y_{n+2}) + \beta\, G(y_n, y_{n+1}, y_{n+1}) + \gamma\, G(y_{n+1}, y_{n+2}, y_{n+2}) + \delta\, G(y_{n+2}, y_{n+3}, y_{n+3})$
$+ \eta \max\{G(y_n, y_{n+1}, y_{n+2}), G(y_n, y_{n+1}, y_{n+1}), G(y_n, y_{n+2}, y_{n+2}), G(y_{n+1}, y_{n+2}, y_{n+2}),$
$G(y_{n+1}, y_{n+1}, y_{n+1}), G(y_{n+1}, y_{n+3}, y_{n+3}), G(y_{n+2}, y_{n+3}, y_{n+3}), G(y_{n+2}, y_{n+1}, y_{n+1}), G(y_{n+2}, y_{n+2}, y_{n+2})\}$

Or

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq \alpha\, G(y_n, y_{n+1}, y_{n+2}) + 2\beta\, G(y_n, y_{n+1}, y_{n+2}) + 2\gamma\, G(y_{n+1}, y_{n+2}, y_{n+3}) + 2\delta\, G(y_{n+1}, y_{n+2}, y_{n+3})$
$+ \eta \max\{G(y_n, y_{n+1}, y_{n+2}), 2G(y_n, y_{n+1}, y_{n+2}), 2\, G(y_n, y_{n+1}, y_{n+2}), 2G(y_{n+1}, y_{n+2}, y_{n+3}),$
$2G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+1}, y_{n+2}, y_{n+3})\}$

Or

$\leq \alpha\, G(y_n, y_{n+1}, y_{n+2}) + 2\beta\, G(y_n, y_{n+1}, y_{n+2}) + 2\gamma\, G(y_{n+1}, y_{n+2}, y_{n+3}) + 2\delta\, G(y_{n+1}, y_{n+2}, y_{n+3})$
$+ \eta \max\{2\, G(y_n, y_{n+1}, y_{n+2}), 2G(y_{n+1}, y_{n+2}, y_{n+3}),$

$\leq (\alpha + 2\beta)\, G(y_n, y_{n+1}, y_{n+2}) + (2\gamma + 2\delta)\, G(y_{n+1}, y_{n+2}, y_{n+3}) + \eta \max\{2G(y_n, y_{n+1}, y_{n+2}),$
$2G(y_{n+1}, y_{n+2}, y_{n+3})\}$

We have following cases:

Case- I: max = $2\, G(y_n, y_{n+1}, y_{n+2})$

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq (\alpha + 2\beta + 2\eta)\, G(y_n, y_{n+1}, y_{n+2}) + (2\gamma + 2\delta)\, G(y_{n+1}, y_{n+2}, y_{n+3})$

$$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq \left(\frac{\alpha+2\beta+2\eta}{1-2\gamma-2\delta}\right) G(y_n, y_{n+1}, y_{n+2})$$

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq G(y_n, y_{n+1}, y_{n+2})$ as $\alpha + \beta + \gamma + \delta + \eta \leq 1/2$

Case- II: max = $2\, G(y_{n+1}, y_{n+2}, y_{n+3})$,

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq (\alpha + 2\beta)\, G(y_n, y_{n+1}, y_{n+2}) + (2\gamma + 2\delta + 2\eta)\, G(y_{n+1}, y_{n+2}, y_{n+3})$

$G(y_{n+1}, y_{n+2}, y_{n+3}) \leq G(y_n, y_{n+1}, y_{n+2})$ as $\alpha + \beta + \gamma + \delta + \eta \leq 1/2$

Similarly now we find

$$G(fx_{n+2}, fx_{n+3}, fx_{n+4}) \le \alpha\, G(gx_{n+2}, gx_{n+3}, gx_{n+4}) + \beta\, G(gx_{n+2}, fx_{n+2}, fx_{n+2}) + \gamma\, G(gx_{n+3}, fx_{n+3}, fx_{n+3}) + \delta\, G(gx_{n+4}, fx_{n+4}, fx_{n+4}) + \eta \max\{G(gx_{n+2}, gx_{n+3}, gx_{n+4}), G(gx_{n+2}, fx_{n+2}, fx_{n+2}), G(gx_{n+2}, fx_{n+3}, fx_{n+3}), G(gx_{n+3}, fx_{n+2}, fx_{n+2}), G(gx_{n+3}, fx_{n+4}, fx_{n+4}), G(gx_{n+4}, fx_{n+2}, fx_{n+2}), G(gx_{n+4}, fx_{n+3}, fx_{n+3})\}$$

Or

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le \alpha\, G(y_{n+1}, y_{n+2}, y_{n+3}) + \beta\, G(y_{n+1}, y_{n+2}, y_{n+2}) + \gamma\, G(y_{n+2}, y_{n+3}, y_{n+3}) + \delta\, G(y_{n+3}, y_{n+4}, y_{n+4}) + \eta \max\{G(y_{n+1}, y_{n+2}, y_{n+3}), G(y_{n+1}, y_{n+2}, y_{n+2}), G(y_{n+1}, y_{n+3}, y_{n+3}), G(y_{n+2}, y_{n+2}, y_{n+2}), G(y_{n+2}, y_{n+4}, y_{n+4}), G(y_{n+3}, y_{n+2}, y_{n+2}), G(y_{n+3}, y_{n+3}, y_{n+3})\}$$

Or

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le \alpha\, G(y_{n+1}, y_{n+2}, y_{n+3}) + 2\beta\, G(y_{n+1}, y_{n+2}, y_{n+3}) + 2\gamma\, G(y_{n+2}, y_{n+3}, y_{n+4}) + 2\delta\, G(y_{n+2}, y_{n+3}, y_{n+4}) + \eta \max\{G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+2}, y_{n+3}, y_{n+4}), 2G(y_{n+2}, y_{n+3}, y_{n+4})\}$$

Or

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le \alpha\, G(y_{n+1}\, y_{n+2}, y_{n+3}) + 2\beta\, G(y_{n+1}, y_{n+2}, y_{n+3}) + 2\gamma\, G(y_{n+2}, y_{n+3}, y_{n+4}) + 2\delta\, G(y_{n+2}, y_{n+3}, y_{n+4}) + \eta \max\{2\, G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+2}, y_{n+3}, y_{n+4})\},$$

$$\le (\alpha+2\beta)\, G(y_{n+1}, y_{n+2}, y_{n+3}) + (2\gamma+2\delta)\, G(y_{n+2}, y_{n+3}, y_{n+4}) + \eta \max\{2G(y_{n+1}, y_{n+2}, y_{n+3}), 2G(y_{n+2}, y_{n+3}, y_{n+4})\}$$

We have following two cases:

Case- I: max = $2\, G(y_{n+1}, y_{n+2}, y_{n+3})$, $G(y_{n+2}, y_{n+3}, y_{n+4}) \le (\alpha+2\beta+2\eta)\, G(y_{n+1}, y_{n+2}, y_{n+3}) + (2\gamma + 2\delta)\, G(y_{n+2}, y_{n+3}, y_{n+4})$

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le \left(\frac{\alpha+2\beta+2\eta}{1-2\gamma-2\delta}\right) G(y_{n+1}, y_{n+2}, y_{n+3})$$

$G(y_{n+2}, y_{n+3}, y_{n+4}) \le G(y_{n+1}, y_{n+2}, y_{n+3})$ as $\alpha + \beta + \gamma + \delta + \eta \le 1/2$

Case-II: max = $2\, G(y_{n+2}, y_{n+3}, y_{n+4})$,

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le (\alpha + 2\beta)\, G(y_{n+1}, y_{n+2}, y_{n+3}) + (2\gamma + 2\delta + 2\eta)\, G(y_{n+2}, y_{n+3}, y_{n+4})$$

$$G(y_{n+2}, y_{n+3}, y_{n+4}) \le \left(\frac{\alpha+2\beta}{1-2\gamma-2\delta-2\eta}\right) G(y_{n+1}, y_{n+2}, y_{n+3})$$

$G(y_{n+2}, y_{n+3}, y_{n+4}) \le G(y_{n+1}, y_{n+2}, y_{n+3})$ as $\alpha + \beta + \gamma + \delta + \eta \le ½$

$$G(y_n, y_n, y_m) \le G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots.. + G(y_{m-1}, y_{m-1}, y_m)$$

$$\le p^n\, G(y_0, y_1, y_2) + p^{n+1}\, G(y_0, y_1, y_2) + \ldots.. + p^{m-1}\, G(y_0, y_1, y_2)$$

$$\le (p^n + p^{n+1} + \ldots.. + p^{m-1})\, G(y_0, y_1, y_2)$$

$$\le \left[\frac{p^n}{1-p}\right] G(y_0, y_1, y_2) \to 0 \text{ as } n \to\infty, \quad p = \alpha + \beta + \gamma + \eta$$

Subsequence $\{fx_n\}$, $\{gx_{n+1}\}$ converge to y. As f (X) is closed subset of g (X) Then there exist $u \in X$ such that gu = y

Now we shall prove that fu = y. So on the contrary Let fu ≠ y.

Then consider on.

$$G(fu, fx_n, fx_n) \le \alpha\, G(gu, gx_n, gx_n) + \beta\, G(gu, fu, fu) + \gamma\, G(gx_n, fx_n, fx_n) + \delta\, G(gx_n, fx_n, fx_n) + \eta \max\{G(gu, gx_n, gx_n), G(gu, fu, gu), G(gu, fx_n, fx_n), G(gx_n, fx_n, fx_n), G(gx_n, fu, fu), G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n) G(gx_n, fu, fu), G(gx_n, fx_n, fx_n)\}$$

$$\le \alpha\, G(y, y, y) + \beta\, G(y, fu, y) + \gamma\, G(y, y, y) + \delta\, G(y, y, y) + \eta \max\{G (y, y, y), G(y, fu, y), G(y, y, y), G(y, y, y), G (y, fu, fu), G(y, y, y), G(y, y, y), G(y, fu, fu), G(y, y, y)\}$$

$$= \beta\, G(y, fu, fu) + \eta \max\{G(fu, y, y)\ G(y, fu, fu)\}$$

$$= \beta\, G(fu, y, y) + 2\eta\, G(fu, y, y)$$

$$= (\beta + 2\eta)\, G(fu, y, y)$$

Hence we get

$(1- \beta - 2\eta)\ \ G(fu, y, y) \leq 0$

$\Rightarrow fu = y$

$\Rightarrow fu = y = gu$. As (f. g) are weakly compatible.

$\Rightarrow gfu = fgu$

$\Rightarrow g\, y = fy$

Now we shall prove that $fy = y$. So on the contrary Let $fy \neq y$. So we can write

$$G(fgx_n, fx_n, fx_n) \leq \alpha\, G(ggx_n, gx_n, gx_n) + \beta\, G(ggx_n, fgx_n, fgx_n) + \gamma\, G(gx_n, fx_n, fx_n) + \delta\, G(gx_n, fx_n, fx_n)$$
$$+ \eta \max\{G(ggx_n, gx_n, gx_n), G(ggx_n, fgx_n, fgx_n), G(ggx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n), G(gx_n, fgx_n, fgx_n), G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n), G(gx_n, fgx_n, fgx_n), G(gx_n, fx_n, fx_n)\}$$

$$\leq \alpha\, G(gy, y, y) + \beta\, G(gy, fy, fy) + \gamma\, G(y, y, y) + \delta\, G(y, y, y) + \eta \max\{G (gy, y, y), G(gy, fy, fy), G(gy, y, y), G(y, y, y), G(y, fy, fy), G(y, y, y), G(y, y, y), G(y, fy, fy), G(y, y, y)\}$$

$$\leq \alpha\, G(fy, y, y) + \eta \max\{G (fy, y, y), G(fy, fy, fy), G(fy, y, y), G(y, fy, fy), G(y, fy, fy)\}$$

$$\leq \alpha\, G(fy, y, y) + 2\eta\, G(fy, y, y)$$

$$= (\alpha + 2\eta)\, G(fy, y, y)$$

Or

$(1 - \alpha - 2\eta)\, G(fy, y, y) \leq 0$

$\Rightarrow fy = y = gy$

Uniqueness:-To prove this part we assume $y_1 \neq y$ is another fixed point of f and g.

Then consider on

$$G(y_1, y, y) = G(fy_1, fy, fy)$$

$$\leq \alpha\, G(gy_1, gy, gy) + \beta\, G(gy, fy, fy) + \gamma\, G(gy, fy, fy) + \delta\, G(gy, fy, fy) + \eta \max\{G(gy_1, gy, gy), G(gy, fy_1, fy_1), G(gy_1, fy, fy), G(gy, fy, fy), G(gy, fy_1, fy_1), G(gy, fy, fy), G(gy, fy, fy)\ G(gy, fy_1, fy_1), G (gy, fy, fy)\}$$

Or

$$G(y_1, y, y) \leq \alpha\, G(y_1, y, y) + \beta\, G(y_1, y_1, y_1) + \gamma\, G(y, y, y) + \delta\, G(y, y, y) + \eta \max\{G(y_1, y, y), G(y, y_1, y)\ G (y_1, y, y), G(y, y, y), G(y, y_1, y), G(y, y, y), G(y, y, y), G(y, y_1, y), G(y, y, y)\}$$

Or

$$G(y_1, y, y) \leq \alpha\, G(y_1, y, y) + \eta \max\{G(y_1, y, y), G(y, y_1, y_1)$$
$$\leq \alpha\, G(y_1, y, y) + \eta \max\{G(y_1, y, y), G(y, y_1, y) + G(y_1, y, y)\}$$
$$= \alpha\, G(y_1, y, y) + \eta \max\{G(y_1, y, y), 2\, G(y_1, y, y)\}$$
$$= (\alpha + 2\eta)\, G(y_1, y, y)$$

Or

$(1- \alpha - 2\eta)\ \ G(y_1, y, y) \leq 0$

$\Rightarrow\ G (y_1, y, y) = 0$

$\Rightarrow\ y_1 = y$

Theorem 2: Let A, B, C, S, R and T be self mappings of a complete G-metric space (X, G) and

(1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, $C(X) \subseteq R(X)$ and A(X) or B(X) or C(X) is a closed subset of X.

(2) $G(Ax, By, Cz) \leq\ a\, G(By, Ax, Rx) + b\, G(Cz, By, Ty) + c\, G(Ax, Cz, Sz) + d \max\{G(Cz, By, Ty) + G(Ty, Ax, Rx) + G(Ax, Cz, Sz), G(Sz, By, Ty) + G(By, Ax, Rx), G (Rx, Cz, Sz)\}$
where $a, b, c, d \geq 0$ and $a + b + 2c + 3d < 1$.

(3) The pairs (A, R), (B, T) and (C, S) are weakly compatible pairs. Then the mappings A, B, C, S, T and R have a common fixed point in X.

Proof: Let $x_0 \in X$ be n arbitrary point By (1) there exist $x_1, x_2, x_3 \in X$ such that $Ax_0 = Tx_1 = y_0$, $Bx_1 = Sx_2 = y_1$ and $Cx_2 = Rx_3 = y_2$ Inductively we get a sequence $(y_n\}$ in X such that

$$Ax_n = Tx_{n+1} = y_n,\ Bx_{n+1} = Sx_{n+2} = y_{n+1},\ Cx_{n+2} = Rx_{n+3} = y_{n+2}, \text{ for } n = 0, 1, 2, 3 \,.....\ \{y_n\}$$

Now we shall prove that the sequence is a Cauchy sequence. Let

$$d_m = G(y_n, y_{n+1}, y_{n+2})$$

Then we have

$$\begin{aligned}
d_n &= G(y_n, y_{n+1}, y_{n+2}) \\
&= G(Ax_n, Bx_{n+1}, Cx_{n+2}) \\
&\le a\,G(Bx_{n+1}, Ax_n, Rx_n) + b\,G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}) + c\,G(Ax_n, Cx_{n+1}, Sx_{n+1}) + d\max\{G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}) \\
&\quad + G(Tx_{n+1}, Ax_n, Rx_n), G(Ax_n, Cx_{n+2}, Sx_{n+2}), G(Sx_{n+2}, Bx_{n+1}, Tx_{n+1}), G(Bx_{n+1}, Ax_n, Rx_n), G(Rx_n, Cx_{n+2}, Sx_{n+2})\} \\
&\le a\,G(y_{n+1}, y_n, y_{n-1}) + b\,G(y_{n+2}, y_{n+1}, y_n) + c\,G(y_n, y_{n+1}, y_n) + d\max\{G(y_{n+2}, y_{n+1}, y_n) + G(y_n, y_n, y_{n-1}) \\
&\quad + G(y_n, y_{n+2}, y_{n+1}), G(y_{n+1}, y_{n+1}, y_n) + G(y_{n+1}, y_n, y_{n-1}), G(y_{n-1}, y_{n+2}, y_{n+1})\} \\
&\le a\,d_{n-1} + bd_n + cd_n + d\max\{d_n + d_{n-1} + d_n, d_n + d_{n-1}, d_n + d_{n-1}\}
\end{aligned}$$

Clearly max will be last term

i.e. $d_n + d_{n-1}$

$d_n \le a\,d_{n-1} + bd_n + Cd_n + d\,(d_n + d_{n-1})$

$d_n \le (a + d)\,d_{n-1} + (b + c + d)\,d_n$

$(1 - b - c - d)\,d_n \le (a + d)\,d_{n-1}$

$$d_n \le \left(\frac{a+b}{1-b-c-d}\right) d_{n-1}$$

So we have

$d_n \le d_{n-1}$

Next we find.

$$\begin{aligned}
d_{n+1} &= G(y_{n+1}, y_{n+2}, y_{n+3}) \\
&= G(Ax_{n+1}, Bx_{n+2}, Cx_{n+3}) \\
&\le a\,G(Bx_{n+2}, Ax_{n+1}, Rx_{n+1}) + b\,G(Cx_{n+3}, Bx_{n+2}, Tx_{n+2}) + c\,G(Ax_{n+1}, Cx_{n+3}, Sx_{n+3}) + d\max\{G(Cx_{n+3}, Bx_{n+2}, Tx_{n+2}), \\
&\quad G(Tx_{n+2}, Ax_{n+1}, Rx_{n+1}), G(Ax_{n+1}, Cx_{n+3}, Sx_{n+3}), G(Sx_{n+3}, Bx_{n+2}, Tx_{n+2}), G(Bx_{n+2}, Ax_{n+2}, Rx_{n+1}), \\
&\quad G(Rx_{n+1}, Cx_{n+3}, Sx_{n+3})\} \\
&\le a\,G(y_{n+2}, y_{n+1}, y_n) + b\,G(y_{n+3}, y_{n+2}, y_{n+1}) + c\,G(y_{n+1}, y_{n+3}, y_{n+2}) + d\max\{G(y_{n+3}, y_{n+2}, y_{n+1})\ G(y_{n+1}, y_{n+1}, y_n), \\
&\quad G(y_{n+1}, y_{n+3}, y_{n+2}), G(y_{n+2}, y_{n+2}, y_{n+1}), G(y_{n+2}, y_{n+1}, y_n), G(y_n, y_{n+3}, y_{n+2})\}
\end{aligned}$$

Thus for every $n \in N$ we have $d_n \le d_{n-1}$. Further we can say $d_n \le p\,d_{n-1}$ where $p = a+b+2c+3d$. Thus we have $d_n = G(y_n, y_{n+1}, y_{n+2}) \le p\,G(y_{n-1}, y_n, y_{n+1}) \le \,.....\, \le p^n\,G(y_0, y_1, y_2)$

And as $G(x, x, y) \le G(x, y, z)$ we have $G(y_n, y_n, y_{n+1}) \le p^n\,G(y_0, y_1, y_2)$

So,

$$\begin{aligned}
G(y_n, y_n, y_m) &\le G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \,.....\, + G(y_{m-1}, y_{m-1}, y_m) \\
&\le p^n\,G(y_0, y_1, y_2) + p^{n+1}\,G(y_0, y_1, y_2) + \,.....\, + p^{m-1}\,G(y_0, y_1, y_2) \\
&\le (p^n + p^{n+1} + \,.....\, + p^{m-1})\,G(y_0, y_1, y_2) \\
&\le \left[\frac{p^n}{1-p}\right] G(y_0, y_1, y_2) \to 0 \text{ as } n \to \infty
\end{aligned}$$

So, the sequence $\{y_n\}$ is Cauchy sequence. in X, let it converge to a point y in X i.e. $\lim_{n\to\infty} y_n = y$. So the subsequences $\{Ax_n\}$, $\{Tx_{n+1}\}$, $\{Bx_{n+1}\}$, $\{Sx_{n+2}\}$, $\{Cx_{n+2}\}$,$\{Rx_{n+3}\}$ all converge to y. As C (x) is Closed Subset of R(X) these exist $u\in X$ such that Ru = y. Now we shall prove that Au = y

Consider

$$G(Au, Bx_{n+1}, Cx_{n+2}) \le a\, G(Bx_{n+1}, Au, Ru) + b\, G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}) + c\, G(Au, Cx_{n+2}, S_{n+2}) + d \max\{G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}),\ G(Tx_{n+1}, Au, Ru), G(Au, Cx_{n+2}, Sx_{n+2}), G(Sx_{n+2}, Bx_{n+1}, Tx_{n+1}), G(Bx_{n+1}, Au, Ru), G(Ru, Cx_{n+2}, Sx_{n+2})\}$$

$$\le a\, G(y, Au, Ru) + b\, G(y, y, y) + c\, G(Au, y, y) + d \max\{G(y, y, y), G(y, Au, Ru), G(Au, y, y), G(y, y, y), G(y, Au, Ru), G(Ru, y, y), G(y, Au, Ru), G(Ru, y, y)\}$$

$$\le a\, G(y, Au, Ru) + c\, G(Au, y, y) + d \max\{G(Au, Ru, y), G(Au, y, y), G(Ru, y, y), \text{ as } Ru = y$$

$$\le a\, G(y, Au, y) + c\, G(Au, y, y) + d \max\{G(Au, y, y), G(Au, y, y), G(y, y, y)\}$$

Or

$G(Au, y, y) \le (a + c + d)\, G(Au, y, y)$

$\Rightarrow (1 - a - c - d)\, G(Au, y, y) \le 0$

$\Rightarrow G(Au, y, y) = 0$

$\Rightarrow Au = y$ as $Ru = y$, $Au = y$

$\Rightarrow Au = Ru = y$

Now we will use the definition of weak compatibility of (R, A)

$\Rightarrow A\, Ru = R\, Au$

$\Rightarrow Ay = Ry$

Now we shall prove Ay = y and on the contrary Let $Ay \ne y$ then we consider on

$$G(Ay, Bx_{n+1}, Cx_{n+2}) \le a\, G(Bx_{n+1}, Ay, Ry) + b\, G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}) + c\, d\, (Ay, Cx_{n+2}, Sx_{n+2}) + d \max\{G(Cx_{n+2}, Bx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Ay, Ry), G(Ay, Cx_{n+2}, Sx_{n+2}), G(Sx_{n+2}, Bx_{n+1}, Tx_{n+1}), G(Bx_{n+1}, Ay, Ry), G(Ry, Cx_{n+2}, Sx_{n+2})\}$$

$$\le a\, G(y, Ay, Ry) + b\, G(y, y, y) + c\, d\, (Ay, y, y) + d \max\{G(y, y, y), G(y, Ay, Ay), G(Ay, y, y), G(y, y, y), G(y, Ay, Ay), G(Ay, y, y)\}$$

$$\le a\, G(y, y, Ay) + c\, G(y, y, Ay) + d\, G(Ay, y, y)$$

$\Rightarrow G(Ay, y, y) \le (a + c + d)\, G(Ay, y, y)$

$\Rightarrow Ay = y = Ry.$

Thus y is common fixed point of A and R. As $y = Ay \subseteq T(X)$ $\exists$ w such that Tv = y. we shall now prove Bv=y

$G(y, Bv, Cx_{n+2}) = G(Ay, Bv, Cx_{n+2})$

$$\le a\, G(Bv, Ay, Ry) + bG(Cx_{n+2}, Bv, Tv) + c\, G(Ay, Cx_{n+2}, Sx_{n+2}) + d\max\{G(Cx_{n+2}, Bv, Tv), G(Tv, Ay, Ry), G(Ay, Cx_{n+2}, Sx_{n+2}), G(Sx_{n+2}, Bv, Tv), G(Bv, Ay, Ry), G(Ry, Cx_{n+2}, Sx_{n+2}),$$

$$\le a\, G(Bv, y, y) + bG(y, Bv, Tv) + c\, G(y, y, y) + d \max\{G(y, Bv, Tv), G(Tv, Ay, Ay), G(y, y, y), G(y, Bv, Tv), G(Bv, y, y), G(Ay, y, y)\}$$

$$\le a\, G(Bv, y, y) + bG(Bv, y, y) + d\, (Bv, y, y)$$

$\Rightarrow (1 - a - b - d)\, G(Bv, y, y) \le 0$

$\Rightarrow Bv = y$

Hence

Bv = y = Tv

By weak compatibility BTv = TBv hence By = Ty

Now we prove By = y. So on the contrary Let By $\neq$ y

$$G(Ay, By, Cx_{n+2}) \leq a\, G(By, Ay, Ry) + bG(Cx_{n+2}, By, Ty) + c\, G(Ay, Cx_{n+2}, Sx_{n+2}) + d \max\{G(Cx_{n+2}, By, Ty), G(Ty, Ay, Ry), G(Ay, Cx_{n+2}, Sx_{n+2}), G(Sx_{n+2}, By, Ty), G(By, Ay, Ry), G(Ry, Cx_{n+2}, Sx_{n+2})\}$$

$$\leq aG(Ty, Ay, Ry) + bG(y, By, y) + G(y, y, y) + d \max\{G(y, By, By), G(y, y, y), G(y, y, y), G(y, y, y), G(By, y, y), G(y, y, y)\}$$

$$G(y, By, y) \leq b\, G(y, By, y) + d\, G(y, By, y)$$

$$\Rightarrow (1 - b - d)\; G(y, By, y) \leq 0$$

Hence By = y = Ty. As y = By $\in$ S (X) there exist w $\in$ X such that y = sw. We now prove that Cw = y.

$$G(y, y, Cw) = G(Ay, By, Cw)$$

$$\leq a\, G(By, Ay, Ry) + b\, G(Cw, By, Ty) + c\, G(Ay, Cw, Sw) + d \max\{G(Cw, By, Ty), G(Ty, Ay, Ry), G(Ay, Cw, Sw), G(Sw, By, Ty), G(By, Ay, Ry), G(Ry, Cw, Sw)\}$$

$$\leq a\, G(y, y, y) + b\, G(Cw, y, y) + c\, G(y, Cw, y) + d \max\{G(Cw, y, y), G(y, y, y), G(y, Cw, y), G(y, y, y), G(y, y, y), G(Cw, y, y),$$

$$\leq (b + c + b)\, G(cw, y, y)$$

Or

$$(1 - b - c - d)\, G(cw, y, y) \leq 0$$

$$\Rightarrow Cw = y$$

$$G(y, y, Cy) = G(Ay, By, Cy)$$

$$\leq a\, G(By, Ay, Ry) + b\, G(Cy, By, Ty) + c\, G(Ay, Cy, Sy) + d \max\{G(Cy, By, Ty), G(Ty, Ay, Ry), G(Ay, Cy, Sy), G(Sy, By, Ty), G(By, Ay, Ry), G(Ry, Cy, Sy)\}$$

$$\leq a\, G(y, y, y) + bG(y, y, y) + c\, G(y, Cy, y) + d \max\{G(Cy, y, y), G(y, y, y), G(y, Cy, y), G(y, y, y), G(y, y, y), G(y, Cy, y)\}$$

$$\leq (b + c + d)\, G(y, y, y)$$

Or

$$G(Cy, y, y)\,(1 - b - c - d) \leq 0$$

$$G(Cy, y, y) = 0$$

Cy = y as Sy = y $\Rightarrow$ Sy = Cy = y i.e. y is a common fixed point of S and C. Thus y will be the common fixed point of A, B, C, S, T and R

i.e

$$Ay = Sy = By = Ty = Cy = Ry = y$$

Now we will prove the uniqueness. Let y' is another fixed point of A, B, C, S, T and R.

$$G(y, y, y' = G(Ay, By, Cy) \leq a\, G(By, Ay, Ry) + b\, G(Cy', By, Ty) + c\, G(Ay, Cy', Sy) + d \max\{G(Cy', By, Ty) + G(Ty, Ay, Ry) + G(Ay, Cy', Sy), G(Sy, By, Ty) + G(By, Ay, Ry), G(Ry, Cy', Sy')\}$$

$$\leq a\, G(y, y, y) + b\, G(y', y, y), + c\, G(y, y', y') + d \max\{G(y', y, y) + G(y, y, y) + G(y, y', y'), G(y', y, y) + G(y, y, y), G(y, y, y')\}$$

$$\leq b\, G(y', y, y) + 2\, c\, G(y', y, y) + d \max\{3\, G(y', y, y), G(y', y, y), 2\, G(y', y, y)\}$$

$$\leq (b + 2\, c)\; G(y', y, y) + 3\, d\, G(y', y, y)$$

Or
$G(y, y, y') \leq (b+ 2c + 3d)\, G(y', y, y)$

$(1- b- 2c - 3d)\; G(y, y, y) \leq 0$

As $a + b + 2c + 3d < 1$

We get $G(y, y, y') \leq 0$

$\Rightarrow\; y = y'$

Hence y is the unique common fixed point of A, B, C, S, T and R.

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