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VECTOR SPACES OVER MATRIX FIELDS

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ABSTRACT

Vector spaces defined over matrix fields are generally missing in mathematical literature. We call these vector spaces as M -vector spaces and provide some examples of M -vector spaces.

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Key-Words: *Block matrices, vector space, matrix field, matrix, M -vector spaces.*

INTRODUCTION

A non empty set V is called a vector space over a field F under addition ($a+b$) and scalar multiplication (αa) if the following conditions hold

1. $(V,+)$ is an Abelian group.
2. $\alpha a \in V$,
3. $\alpha(a+b) = \alpha a + \alpha b$,
4. $(\alpha + \beta)a = \alpha a + \beta a$,
5. $(\alpha\beta)a = \alpha(\beta a)$,
6. $1a = a$, $\forall\, a,b \in V$; $\forall\ \alpha, \beta \in F$. Here 1 is the multiplicative identity of the field F .

In the above definition F is an arbitrary field. Therefore we may consider F as a matrix field for the purpose of this article. The addition in V is known as vector addition. An element of V is called a vector and an element of F is called a scalar.

In the textbooks of modern algebra or linear algebra (one may refer [1], [2], [3]) one can find several examples of a vector space. However examples of a vector space defined over a matrix field are generally not found in textbooks.

We consider different matrix fields and provide some examples of a vector space defined over matrix fields and we call these vector spaces as M -vector spaces.

SOME EXAMPLES OF VECTOR SPACES OVER INFINITE MATRIX FIELDS

Let Q, R and C denote the set of all rational, real and complex numbers respectively. Corresponding to these sets we consider three sets namely M_Q, M_R and M_C respectively. These sets are given by

$$M_Q = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in Q \right\},$$

$$M_R = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in R \right\},$$

$$M_C = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in C \right\}.$$

One can easily verify that M_Q, M_R and M_C satisfy field axioms and hence these are fields (one may refer [4]) such that $M_Q \subseteq M_R \subseteq M_C$. Since every field can be regarded as a vector space over itself. Therefore M_Q is a vector space over M_Q, M_R is a vector space over M_R and M_C is a vector space over M_C. In addition M_R is a vector space over M_Q and M_C is a vector space over M_R.

We now consider some other examples.

Let $F_1 = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in Q \right\}$,

$$F_2 = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in R \right\},$$

$$F_3 = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in C \right\},$$

$$F_4 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Q \right\},$$

$$F_5 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R \right\},$$

$$F_6 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in C \right\}.$$

In the same line as mentioned above it is easy to see that each one of the above sets is a vector space over itself. Further F_2 is a vector space over F_1, F_3 is a vector space over F_2, F_5 is a vector space over F_4 and F_6 is a vector space over F_5. One may construct many other examples of M -vector spaces.

Let $F_Q = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a \in Q \right\}$,

$$F_R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a \in R \right\} \text{ and}$$

$$F_C = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a \in C \right\}.$$

Then one can verify that F_Q is a vector space over M_Q; F_R is a vector space over M_Q and M_R; and F_C is a vector space over M_Q, M_R and M_C.

We conclude this section by providing few more examples. We shall consider examples of a vector space of all square matrices of order two in this context.

Let $M_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Q \right\}$,

$$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \right\}, \text{ and}$$

$$M_3 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in C \right\}.$$

One may verify that M_1 is a vector space over M_Q, M_2 is a vector space over M_R and M_Q; M_3 is a vector space over M_C, M_R, and M_Q. We can also see that M_1 is a vector space over F_Q, M_2 is a vector space over F_Q and F_R; and M_3 is a vector space over F_Q, F_R and F_C.

In all of the above examples vector addition is defined as usual matrix addition and scalar multiplication is defined as usual matrix multiplication.

SOME EXAMPLES OF VECTOR SPACES OVER FINITE MATRIX FIELDS

Some matrix representations of a finite field of order p are given in [4]. Using these fields one can obtain different examples of a vector space over a finite matrix field for each $p > 0$. Let (refer [5])
$$V_M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \right\}$$
then V_M is a vector space over V_M. Also if we take $F_M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$ then it is easy to verify that V_M is a vector space over F_M. In these examples vector addition is defined as matrix addition modulo $p(p = 3\ for\ V_M)$ and scalar multiplication is defined as matrix multiplication modulo $p(p = 3\ for\ V_M)$. Similarly one can obtain several examples.

CONCLUDING REMARKS

It may be noted that the theory of M -vector spaces seems almost analogous to the theory of usual vector spaces.

One can study various concepts of linear algebra (like linear combination, linear dependence, linear independence, and basis etc.) for M -vector spaces also. By considering a vector space of n -tuples of matrices one can define different liner transformations and find matrix representations of a liner transformation and linear operator. Similarly one can determine eigenvalues of a linear operator. In this case matrix of a linear transformation (or linear operator) will be a block matrix and eigenvalues of a given linear operator will be a matrix.

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