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STUDY ON SOME PROPERTIES OF SKEW-SYMMETRIC AND SKEW-CIRCULANT MATRIX OVER SKEW FIELD

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ABSTRACT

In this paper, some properties of skew-symmetric and skew-circulant matrix were extended from general complex domain to skew field. The relationships between skew-symmetric and skew-circulant matrix and symmetric circulant matrix, symmetric and skew-circulant matrix and skew-circulant matrix, over skew field, were obtained. Meanwhile, the linear expression of skew-symmetric and skew-circulant matrix under fundamental skew-circulant matrix was also obtained. In addition, over skew field, the sufficient condition, which could infer that one matrix was skew-symmetric and skew-circulant matrix and that skew-symmetric and skew-circulant matrices were commutative, as well as some properties of inverse matrix of skew-symmetric and skew-circulant matrix were acquired.

Keywords: *Skew-symmetric and Skew-circulant Matrix; Symmetric and Skew-circulant Matrix; Skew-circulant Matrix.*

1. INTRODUCTION

A momentous component of matrix theory, circulant matrix, is a kind of important special matrix, and it is also one of the most important and active research fields of applied mathematics. From when the concept of circulant matrix was proposed to the present, there were a series of significant achievements about circulant matrices. For example, in 2009, Li Tianzeng did some research on circulant matrix [2] and in 2012, Dai Jiejie did some research on symmetric circulant matrix[3]. In 2013 and 2014, Jiang Jiaqing got some results about skew-circulant matrix[4] and symmetric and skew-circulant matrix[5]. Correspondingly, over skew field, the concepts on circulant matrix, skew-circulant matrix, symmetric circulant matrix and symmetric and skew-circulant matrix, can also be defined. However, in this paper, skew-symmetric and skew-circulant matrix over skew field was mainly discussed.

In the references [6], [7], [8], the authors have made a comprehensive study on the skew-symmetric and skew-circulant matrix in general complex domain. In the reference [6], the diagonalization of skew-symmetric and skew-circulant matrix was discussed by using Vandermonde matrix, through which some results were given. In the reference [7], based on the special characters of this kind of matrices, the particular laws and special characters of skew-symmetric and skew-circulant matrices' eigenvalues and eigenvectors were analyzed by using Fourier matrix and Vandermonde matrix. Moreover, some matrix inversion techniques of skew-symmetric and skew-circulant matrix were researched. In the reference [8], block skew- symmetric and skew-circulant matrix was discussed, and the eigenvalues of block skew-symmetric and skew-circulant matrices whose sub-block were special circulant matrices and their matrices inversion were studied. However, based on the references upon, in this paper, skew-symmetric and skew-circulant matrix over skew field have been researched and some of its properties were extended from general complex domain to skew field, directly or indirectly. The relationships between skew-symmetric and skew-circulant matrix and special circulant matrices over skew field were given. Furthermore, the linear expression, the property of commutative and the sufficient condition of skew-symmetric and skew-circulant matrix were also obtained.

In this paper, let K be a skew field, $K^{m\times n}$ represents the set of the unit $m\times n$ matrix, $M_n(K)$ represents the set of the unit $n\times n$ matrix and I_n is $n\times n$ identity matrix over skew field. $C(a_0,a_1,a_2,\cdots,a_{n-1})$ means n-order circulant matrix, $C_{-1}(a_0,a_1,a_2,\cdots,a_{n-1})$ means n-order skew-circulant matrix, $SC(a_0,a_1,a_2,\cdots,a_{n-1})$ means n-order symmetric circulant matrix and $SC_{-1}(a_0,a_1,a_2,\cdots,a_{n-1})$ means n-order symmetric and skew-circulant matrix, over skew field.

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Definition 1: Let $A = C_{-1}(a_0, a_1, a_2, \cdots, a_{n-1})$ be a skew-circulant matrix over skew field and it meets $A^T = -A$, then A is called skew-symmetric and skew-circulant matrix over skew field.

From the definition, if A is skew-symmetric and skew-circulant matrix, then $a_0 = 0, a_i = a_{n-i}, (i = 1, 2, \cdots, n-1)$. Thus, when $n = 2p, (p \in N)$,

$$A = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_p & -a_{p-1} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

which can be denoted briefly as $A = S_{-1}C_{-1}(0, a_1, a_2, \cdots, a_{p-1}, a_p, a_{p-1}, \cdots, a_1)$, and when $n = 2p+1, (p \in N)$,

$$A = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_p & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_{p-2} & -a_{p-3} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

which can be denoted as $A = S_{-1}C_{-1}(0, a_1, a_2, \cdots, a_p, a_p, a_{p-1}, \cdots, a_1)$ briefly.

Definition 2: In the group of circulant matrices over skew field, matrices

$$\pi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

are called fundamental circulant matrix and fundamental skew-circulant matrix respectively, among which 1 is the identity element over skew field and -1 is negative element of identity element.

Lemma: For n-order fundamental skew-circulant matrix η over skew field, the following statements are tenable:

$$\eta^n = -I, \eta^T = -\eta^{n-1}, \eta\eta^T = I.$$

2. MAIN CONCLUSIONS

Theorem 1: If $A, B, F \in M_n(K)$, $A = S_{-1}C_{-1}(0, a_1, a_2, \cdots, a_p, \cdots, a_1)$, $F \in SC(0, 0, \cdots, 1)$, and $B = SC_{-1}(a_1, a_2, \cdots, a_p, \cdots, a_1, 0$, then $AF = B$.

Proof: When $n = 2p$, if

$$F = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$A=\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_p & -a_{p-1} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

then

$$AF=\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_1 & 0 \\ a_2 & a_3 & a_4 & \cdots & a_p & a_{p-1} & a_{p-2} & \cdots & 0 & -a_1 \\ a_3 & a_4 & a_5 & \cdots & a_{p-1} & a_{p-2} & a_{p-3} & \cdots & -a_1 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & 0 & -a_1 & \cdots & -a_{p-3} & -a_{p-2} & -a_{p-1} & \cdots & -a_3 & -a_2 \\ 0 & -a_1 & -a_2 & \cdots & -a_{p-2} & -a_{p-1} & -a_p & \cdots & -a_2 & -a_1 \end{pmatrix}$$

$$= B = SC_{-1}(a_1, a_2, \cdots, a_p, \cdots, a_1, 0$$

Similarly, when $n = 2p+1$, the conclusion can also be proved.

Corollary: If $A \in M_n(K)$, then $A = S_{-1}C_{-1}(0, a_1, a_2, \cdots, a_p, \cdots, a_1)$ is nonsingular if and only if $B = SC_{-1}(a_1, a_2, \cdots, a_p, \cdots, a_1, 0$ is nonsingular.

Theorem 2: If $A, T \in M_n(K)$, $A = S_{-1}C_{-1}(a_0, a_1, a_2, \cdots, a_p, \cdots, a_1)$ is skew- symmetric and skew-circulant matrix and $T = SC_{-1}(1, 0, 0, \cdots, 0)$ is symmetric and skew-circulant matrix, then

$$TA = SC_{-1}(0, a_1, a_2, \cdots, a_p, \cdots, a_1),$$

$$AT = SC_{-1}(0, -a_1, -a_2, \cdots, -a_p, \cdots, -a_2, -a_1),$$

And $TA = -AT$.

Proof: When $n = 2p+1$, let

$$T=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$A=\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_p & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_{p-2} & -a_{p-3} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

Then

$$TA=\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_p & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_p & a_{p-1} & a_{p-2} & \cdots & a_1 & 0 \\ a_2 & a_3 & a_4 & \cdots & a_{p-1} & a_{p-2} & a_{p-3} & \cdots & 0 & -a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_1 & 0 & \cdots & -a_{p-2} & -a_{p-1} & -a_p & \cdots & -a_4 & -a_3 \\ a_1 & 0 & -a_1 & \cdots & -a_{p-1} & -a_p & -a_{p-1} & \cdots & -a_3 & -a_2 \end{pmatrix}$$

$$=SC_{-1}(0,a_1,a_2,\cdots,a_p,\cdots,a_1)$$

$$AT=\begin{pmatrix} 0 & -a_1 & -a_2 & \cdots & -a_p & -a_p & -a_{p-1} & \cdots & -a_2 & -a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_{p-2} & -a_{p-3} & \cdots & 0 & a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

$$=SC_{-1}(0,-a_1,-a_2,\cdots,-a_p,\cdots,-a_1)$$

Therefore, $TA=-AT$.

When $n=2p$, the proof method is similar.

Theorem 3: Let $A,B\in M_n\left(K\right)$,if $A=S_{-1}C_{-1}(0,a_1,a_2,\cdots,a_p,a_p,a_{p-1},\cdots,a_2,a_1)$ is 2n+1-order symmetric and skew-circulant matrix, then

$$A\eta=\eta A=B=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)$$

among which B means 2n+1-order skew-circulant matrix which the first row elements is $(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)$ and η is 2n+1-order fundamental skew-circulant matrix.

Proof: If

$$A=\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_p & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_{p-2} & -a_{p-3} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

$$\eta=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

then

$$A\eta=\begin{pmatrix} -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ -a_3 & -a_2 & -a_1 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_5 & a_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \\ 0 & -a_1 & -a_2 & \cdots & -a_p & -a_p & -a_{p-1} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

$$=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)=B=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)$$

$$\eta A=\begin{pmatrix} -a_1 & 0 & a_1 & \cdots & a_{p-1} & a_p & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_4 & a_3 \\ -a_3 & -a_2 & -a_1 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_5 & a_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \\ 0 & -a_1 & -a_2 & \cdots & -a_p & -a_p & -a_{p-1} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

$$=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)=B=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)$$

Thus, $A\eta=\eta A=B=C_{-1}(-a_1,0,a_1,\cdots,a_{p-1},a_p,a_p,\cdots,a_3,a_2)$.

Theorem 4: Every n-order skew-circulant matrix $A=C_{-1}(a_0,a_1,a_2,\cdots,a_{n-1})$ over skew field can be expressed as the sum of a specific symmetric and skew-circulant matrix $B=SC_{-1}(b_0,b_1,b_2,\cdots,b_{p-1},b_p,-b_{p-1},\cdots,-b_1)$ and a skew-symmetric and skew- circulant matrix $C=S_{-1}C_{-1}(0,c_1,c_2,\cdots,c_{p-1},c_p,c_{p-1},\cdots,c_1)$ over skew field.

Proof: From what is known, let $b_i=\frac{1}{2}\left(a_i+a_{n-i}\right),c_i=\frac{1}{2}\left(a_i-a_{n-i}\right)$, among which $i=0,1,2,\cdots,n-1,$ then $B+C=A$.

Theorem 5: $A\in M_n(K)$ is skew-symmetric and skew-circulant matrix if and only if A can be expressed by $\eta^0=I,\eta,\eta^2,\cdots,\eta^{n-1}$, $A=f(\eta)$,among which η is n-order fundamental skew-circulant matrix over skew field. When $n=2p$, $f(\eta)=a_1\eta+a_2\eta^2+\cdots+a_{p-1}\eta^{p-1}+a_p\eta^p+a_{p-1}\eta^{p+1}+\cdots+a_2\eta^{n-2}+a_1\eta^{n-1}$ and when $n=2p+1$,

$$f(\eta)=a_1\eta+a_2\eta^2+\cdots+a_{p-1}\eta^{p-1}+a_p\eta^p+a_p\eta^{p+1}+a_{p-1}\eta^{p+2}+\cdots+a_2\eta^{n-2}+a_1\eta^{n-1}$$

Proof: When $n=2p$

1) **Sufficiency:** If $A=f(\eta)$,

$f(\eta)=a_1\eta+a_2\eta^2+\cdots+a_{p-1}\eta^{p-1}+a_p\eta^p+a_{p-1}\eta^{p+1}+\cdots+a_2\eta^{n-2}+a_1\eta^{n-1}$, then

$$A=\begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & 0 & \cdots & 0 \end{pmatrix}+\cdots+$$

$$\begin{pmatrix} 0 & 0 & \cdots & a_p & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{p-1} & 0 & \cdots & 0 & \cdots & a_p \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -a_{p-1} & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & a_1 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_p & -a_{p-1} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

Because of the Definition 1, the sufficient condition can be proved.

2) Necessity:

$$A = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{p-1} & a_p & a_{p-1} & \cdots & a_2 & a_1 \\ -a_1 & 0 & a_1 & \cdots & a_{p-2} & a_{p-1} & a_p & \cdots & a_3 & a_2 \\ -a_2 & -a_1 & 0 & \cdots & a_{p-3} & a_{p-2} & a_{p-1} & \cdots & a_4 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_2 & -a_3 & -a_4 & \cdots & -a_{p-1} & -a_p & -a_{p-1} & \cdots & 0 & a_1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_2 & 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots +$$

$$\begin{pmatrix} 0 & 0 & \cdots & a_p & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{p-1} & 0 & \cdots & 0 & \cdots & a_p \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -a_{p-1} & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & a_1 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -a_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= f(\eta) = a_1\eta + a_2\eta^2 + \cdots + a_{p-1}\eta^{p-1} + a_p\eta^p + a_{p-1}\eta^{p+1} + \cdots + a_2\eta^{n-2} + a_1\eta^{n-1}$$

When $n = 2p+1$, the method of proof is completely consistent.

Theorem 6: Let $A, B \in M_n(K)$ be skew-symmetric and skew-circulant matrices. If the element a_i of A and the element b_j of B are commutative, then A and B are commutative, too. Moreover, AB and BA are all skew-symmetric and skew-circulant matrices over skew field.

Proof: From Theorem 5, if $A, B \in M_n(K)$ are all skew-symmetric and skew- circulant matrices $(n = 2p)$, then

$$A = a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots + a_p\eta^p + \cdots + a_1\eta^{n-1} = f(\eta) = \sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j}$$

$$B = b_1\eta + b_2\eta^2 + b_3\eta^3 + \cdots + b_p\eta^p + \cdots + b_1\eta^{n-1} = g(\eta) = \sum_{i=1}^{p} b_i\eta^i + \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j}$$

$$\begin{aligned} AB &= f(\eta)g(\eta) \\ &= \left(\sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j})(\sum_{i=1}^{p} b_i\eta^i + \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j} \right) \\ &= \sum_{i=1}^{p} a_i\eta^i \sum_{i=1}^{p} b_i\eta^i + \sum_{i=1}^{p} a_i\eta^i \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j} + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \sum_{i=1}^{p} b_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j} \\ &= \sum_{i=1}^{p} b_i\eta^i \left(\sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \right) + \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j} \left(\sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \right) \\ &= \left(\sum_{i=1}^{p} b_i\eta^i + \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j} \right)\left(\sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \right) \\ &= \left(\sum_{i=1}^{p} b_i\eta^i + \sum_{j=1}^{p-1} b_{p-j}\eta^{p+j})(\sum_{i=1}^{p} a_i\eta^i + \sum_{j=1}^{p-1} a_{p-j}\eta^{p+j} \right) \\ &= g(\eta)f(\eta) = BA = h(\eta) \end{aligned}$$

From the Lemma, $\deg h(\eta) \le n-1$, therefore AB and BA are all skew-symmetric and skew-circulant matrices over skew field.

When $n = 2p+1$, the proof method is similar.

Theorem 7: Let $A \in M_n(K)$ and A is skew-circulant matrix. If for arbitrary $X \in K^n$, $X^T AX = 0$, then A is skew-symmetric and skew-circulant matrix.

Proof: From what is given, let $X_i = (0 \cdots 1 \cdots 0)^T$, $X_{ij} = (0 \cdots 1 \cdots 1 \cdots 0)^T$, among which all the elements of X_i is 0 except its i th row element is 1;all the elements of X_{ij} is 0 except its i th and j th row elements are 1.

If $X_i^T AX_i = 0$, and that is

$$(0 \cdots 1 \cdots 0)\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & a_0 \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = 0,$$

Then $a_0 = 0$.

If $X_{ij}^T AX_{ij} = 0$, and that is

$$\left(0\cdots1\cdots1\cdots0\right)\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & a_0 \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = 0$$

which is equivalent with $\left(0\cdots1\cdots1\cdots0\right)\begin{pmatrix} a_{i-1}+a_{j-1} \\ a_{i-2}+a_{j-2} \\ \vdots \\ -a_i - a_j \end{pmatrix} = 0$,so $a_0 + a_{j-i} + a_0 + a_{i-j} = 0$.

From $a_0 = 0$, $a_{j-i} = -a_{i-j}, i, j = 0,1,2,\cdots,n-1$ is known, thus $A^T = -A$ .Above with A is skew-circulant matrix, then A is skew-symmetric and skew-circulant matrix.

Theorem 8: If $A \in M_n(K)$ is nonsingular skew-symmetric and skew-circulant matrix, then A^{-1} is also nonsingular skew-symmetric and skew-circulant matrix and $A^{-1} = S_{-1}C_{-1}(0, x_1, x_2, \cdots, x_p, \cdots, x_1)$, among which $X = (0, x_1, x_2, \cdots, x_p, \cdots, x_1)^T$ is the solution of right linear system of equations $AX = e_n$, $e_n = (0, 0, \cdots, 0, 1)^T$.

Proof: Let $B = S_{-1}C_{-1}(0, x_1, x_2, \cdots, x_p, \cdots, x_1)$ .If $n = 2p$, then

$$\begin{aligned} AB &= (a_1\eta + a_2\eta^2 + \cdots a_p\eta^p + \cdots + a_1\eta^{n-1})(x_1\eta + x_2\eta^2 + \cdots x_p\eta^p + \cdots + x_1\eta^{n-1}) \\ &= (-a_1x_1 - a_2x_2 - \cdots - a_px_p - \cdots - a_1x_1)\eta^0 + (-a_2x_1 - \cdots - a_px_{p-1} - \cdots - a_1x_2)\eta \\ &\quad + \cdots + (a_1x_2 + a_2x_3 + \cdots + a_px_{p-1} + \cdots + a_2x_1)\eta^{2p-1} \end{aligned}$$

However, $X = (0, x_1, x_2, \cdots, x_p, \cdots, x_1)^T$ is the solution of right linear system of equations $AX = e_n$,so

$$\begin{cases} a_1x_1 + a_2x_2 + \cdots + a_px_p + \cdots + a_1x_1 = -1 \\ a_2x_1 + a_3x_2 + \cdots a_px_{p-1} + \cdots + a_1x_2 = 0 \\ \cdots\cdots \\ a_1x_2 + a_2x_3 + \cdots a_px_{p-1} + \cdots + a_2x_1 = 0 \end{cases}$$

Obviously, $AB = \eta^0 = I_n$, therefore, $A^{-1} = B = S_{-1}C_{-1}(0, x_1, x_2, \cdots, x_p, \cdots, x_1)$.

In like manner, when $n = 2p+1,$ the conclusion is also correct.

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