International Research Journal of Pure Algebra-6(4), 2016, 285-287
Available online through www.rjpa.info ISSN 2248–9037

NILPOTENCY OF IDEALS GENERATED BY SETS CONTAINED IN THE CENTER

[1]M. MANJULA DEVI*, [2]K. SUVARNA

[1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

(Received On: 20-04-16; Revised & Accepted On: 30-04-16)

ABSTRACT

In this paper we consider R be a nonassociative and noncommutative ring. Let S be an additive subgroup of R such that (S, R) = 0. Now we take V={x∈R/ (x, y) = 0, for all y∈R}. From (S, R)=0, it follows that s∈V, where s is in S, and V is subring of R. Using these we show that V equals the center C of R, the set I=S+SR is an ideal of R and $(S+SR)^n = S^n+S^nR$ for all positive integers n. Also it is proved that the ideal of R generated by S is nilpotent if and only if the subring generated by S is nilpotent.

***Mathematics subject classification**: Primary 17A30.*

***Keywords:** Associator, Commutator, Nucleus, Center, Nilpotent ideal.*

INTRODUCTION

Yen and Hentzel [3] studied the nonassociative rings with the ideal generated by sets contained in two of the three nuclei. In this paper we consider R be a nonassociative and noncommutative ring. The associator (a, b, c) and commutator (x, y) are defined by (a, b, c) = (ab)c–a(bc), (x, y)=xy-yx for all a, b, c, x, y in R. The nucleus N and center C of R are defined by N={n∈R/(n, R, R}= (R, n, R)=(R, R, n=0} and center C={c∈N/(c, R)=0}. The ideal of R is nilpotent if there is a positive integer n such that every product involving n elements is zero. In any nonassociative ring we have the Teichmuller identity (ab, c, d)-(a, bc, d)+(a, b, cd)=a(b, c, d)+(a, b, c)d. Thus (R, R, R) + (R, R, R)R=(R, R, R)+R(R, R, R). Kleinfeld [1] showed that (R, R, R)+(R, R, R)R is an ideal of R. This is called the associator ideal. It is the ideal which is generated by all associators. Similarly, we have (R, R)+(R, R)R=(R, R)+R(R, R). Let S be an additive subgroup of R such that (S, R) = 0. So S+SR=S+RS. Examples of S are (R, R) and (R, R, R). Now we take V={x∈R/ (x, y) = 0, for all y∈R}. From (S, R) = 0, it follows that s∈V where s is in S and V is a subring of R. Using these we show that V equals the center C of R. Thus S is contained in the center C of R. Then we prove that the set I=S+SR is an ideal of R and $(S+SR)^n=S^n+S^nR$ for all positive integers n. Also we show that the ideal of R generated by S is nilpotent if and only if the subring generated by S is nilpotent.

PRELIMINARIES

Let R be a nonassociative and noncommutative ring. Let S be an additive subgroup of R such that

$$(S, R)=0 \tag{1}$$

Now we take V={x∈R/(x, y) = 0, for all y∈R}. From (1) it follows that s∈V where s is in S and V is a subring of R. We now prove the following lemmas.

Lemma 1: The set W= {s/s∈V, Rs⊂V} is an ideal of R such that (x, y, s)∈W and (s, y, x)∈W, for s∈V and all x, y∈R.

Proof: From (1), we see that W is a two sided ideal of R. From the Teichmuller identity
a(b, c, d)+(a, b, c)d = (ab, c, d)-(a, bc, d)+(a, b, cd), which holds in any ring, we get z(x, y, s)=(zx, y, s) -(z, xy, s) +(z, x, ys)-(z, x, y)s∈V, since V is a subring of R and (1) holds. Similarly we get z(s, y, x) = (s, y, x)z∈V. Hence (x, y, s)∈W and (s, y, x)∈W.

Corresponding Author: [1]M. Manjula Devi, [1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.*

Lemma 2: Let R be a ring without non zero ideals $\neq$ R satisfying (S, R) = 0. Then V equals the center C of R.

Proof: The ideal W of lemma 1 is contained in V of R. Since R has no non trivial ideal either W=R or W = 0. If W = R, then R is commutative, which is a contradiction to our assumption. Hence W $\neq$ R. So W=0. Then from lemma1 we get (x, y, s) = 0 and (s, y, x) = 0, for all $s\in S$, $x, y\in R$. We know the semijacobi identity (x, z, y)=(x, y, z)+(z, x, y)-(xy, z) +(x, z)y-x(y, z), which holds in any ring, we get (x, s, y) = (x, y, s)+(s, x, y)-(xy, s)+(x, s)y-x(y, s) = 0, from (1), (x, y, s) = 0 and (s, y, x)=0. Hence S contained in the nucleus N of R. Therefore V equals the center C of R.

MAIN RESULTS

From lemma 2 we have that S is contained in the center C of R. Let the set I=S+SR. From (1) we have S+SR=S+RS. By assumption $SR\subset I$ and $Rs\subset I$.

Lemma 3: If S is an additive subgroup of R such that (S, R) = 0, then the set I=S+SR is a two sided ideal of R.

Proof: Since S is in the center of R, RI=R(S+SR)=RS+R.SR=RS+RS. $R\subset I+IR$ and IR=(S+RS)R=SR+RS.R=SR+R.SR $\subset$ I+RI. Hence I+IR=I+RI. Now IR=(S+SR)R=SR+SR.R =$SR+SR^2\subset I$. So I is a right ideal of R. Since I+IR=I+RI, we have that I is a left ideal of R. Hence I is an ideal of R.

Lemma 4: If S is an additive subgroup of R such that (S, R)=0, then $S^n+S^nR=S^n+RS^n$ for all positive integers n.

Proof: $\sum_{i=1}^{\infty} S^i$ is an associative subring contained in the center of R. So $(S^i, S^j, R) = (S^i, R, S^j) = (R, S^i, S^j) = 0$ for all integers i, j $\geq$ 1. By induction, it is true for n=1. We assume the result for n. Then we get $S^{n+1}R=S^nS.R=S^n.SR\subset S^n(S+RS)=S^{n+1}+S^n.RS=S^{n+1}+S^nR.S\subset S^{n+1}+(S^n+S^nR)S=S^{n+1}+ S^nR.S =S^{n+1}+RS^n.S=S^{n+1}+R.S^{n+1}$ and $RS^{n+1}=R.S^nS=RS^n.S\subset (S^n+S^nR)S =S^{n+1}+ S^nR.S=S^{n+1} +S^n.RS \subset S^{n+1} +S^n.(S+SR) = S^{n+1}+S^n.SR=S^{n+1}+S^{n+1}.R$.

So, $S^{n+1}+S^{n+1}R=S^{n+1}+RS^{n+1}$, for all positive integers n. This proves the lemma.

Lemma 5: If S is an additive subgroup of R such that (S, R) = 0, then $S^n+S^nR=S^n+RS^n$ is the ideal of R generated by S^n for all positive integers n.

Proof: From lemma 4, we have $S^n+S^nR=S^n+RS^n$. If we replace S by S^n in lemma 3, we get the ideal $S^n+S^nR=S^n+RS^n$, where n is any positive integer.

Lemma 6: If S is an additive subgroup of R such that (S, R) = 0, then $S^iR.S^jR\subset S^{i+j}+S^{i+j}R$ for all integers i, j $\geq$ 1.

Proof: Since S is in the center C of R, by lemma 5, $S^iR.S^jR=S^i.R(S^jR)\subset S^i.(S^jR)\subset S^i(S^j+S^jR)=S^{i+j}+S^i(S^jR)=S^{i+j}+(S^i.S^j).R=S^{i+j}+S^{i+j}R$.

Lemma 7: If S is an additive subgroup of R such that (S, R) = 0, then $S^iR.S^j\subset S^{i+j}+S^{i+j}R$ for all integers i, j $\geq$ 1.

Proof: Since S is in the center C of R and using lemma 5, $S^iR.S^j=S^i.RS^j\subset S^i(S^j+S^jR) =S^{i+j}+S^i.S^jR = S^{i+j}+S^{i+j}.R$.

Lemma 8: If S is an additive subgroup of R such that (S, R) = 0, then $(S+SR)^n = S^n+S^nR$ for all positive integers n.

Proof: We assume the result for all positive integers m $\leq$ n. Then using this inductive hypothesis, the lemma 7 and lemma 6 for all integers i and j, I $\leq$n and j$\leq$n, we get $(S+SR)^{i+j}=(S+SR)^i (S+SR)^j=(S^i+S^iR) (S^j+S^jR) =S^{i+j}+S^i.S^jR+ S^iR.S^j+S^iR.S^jR = S^{i+j}+S^iS^j.R+S^iR.S^j+S^iR.S^jR=S^{i+j}+S^{i+j}R$.

Therefore $(S+SR)^{i+j} = S^{i+j}+S^{i+j}R$.

This proves the lemma

Now we prove the following theorem.

Theorem: Let R be a non associative, non commutative ring and S be an additive subgroup of R such that (S, R) = 0. The ideal of R generated by S is nilpotent if and only if the subring generated by S in nilpotent.

Proof: If the ideal of R generated by S is nilpotent then $(S+SR)^n = 0$. From lemma 8 it follows that $S^n+S^nR = 0$. So the subring generated by S is nilpotent. The converse follows similarly.

© 2016, RJPA. All Rights Reserved

REFERENCES

1. Kleinfeld.E., A class of rings which are very nearly associative, American Mathematical Monthly, Vol.93 (1986), 720-722.
2. Thedy.A., On rings satisfying [(a, b, c), d] = 0, Proceedings of The American Mathematical Society, Vol.29, No.2, July 1971, 250-254.
3. Yen. C.T. and Hentzel.I.R., The ideal generated by sets contained in the nuclei, Soochow Journal of Mathematics, Vol.22, No.3, July (1996), 435-438.

Source of Support: Nil, Conflict of interest: None Declared

[Copy right © 2016, RJPA. All Rights Reserved. This is an Open Access article distributed under the terms of the International Research Journal of Pure Algebra (IRJPA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]