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INVARIANT SUBMANIFOLD OF (4, 1) STRUCTURE MANIFOLD

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ABSTRACT

In this paper, we have studied various properties of a (4, 1) structure manifold and its invariant submanifold. Under two different assumptions, the nature of induced structure ψ has also been discussed.

Keywords: *Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.*

1. INTRODUCTION

Let V^m be a C^∞ m-dimensional Riemannian manifold imbedded in a C^∞ n-dimensional Riemannian manifold M^n, where $m<n$. The imbedding being denoted by

$$f : V^m \longrightarrow M^n$$

Let B be the mapping induced by f i.e. $B=df$

$$df : T(V) \longrightarrow T(M)$$

Let $T(V, M)$ be the set of all vectors tangent to the submanifold $f(V)$. It is well known that

$$B: T(V) \longrightarrow T(V,M)$$

Is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$, which we shall denote by $N(V,M)$. We call $N(V,M)$ the normal bundle of V^m. The vector bundle induced by f from $N(V,M)$ is denoted by $N(V)$. We denote by $C: N(V) \longrightarrow N(V,M)$ the natural isomorphism and by $\eta_s^r(V)$ the space of all C^∞ tensor fields of type (r, s) associated with N (V). Thus $\zeta_0^0(V) = \eta_0^0(V)$ is the space of all C^∞ functions defined on V^m while an element of $\eta_0^1(V)$ is a C^∞ vector field normal to V^m and an element of $\zeta_0^1(V)$ is a C^∞ vector field tangential to V^m.

Let $\bar{X}$ and $\bar{Y}$ be vector fields defined along $f(V)$ and $\tilde{X}, \tilde{Y}$ be the local extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $\left[\tilde{X}, \tilde{Y}\right]$ is a vector field tangential to M^n and its restriction $\left[\tilde{X}, \tilde{Y}\right]/f(V)$ to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $\left[\bar{X}, \bar{Y}\right]$ is defined as

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(1.1) $\left[\bar{X}, \bar{Y}\right] = \left[\tilde{X}, \tilde{Y}\right] \big/ f(V)$

Since B is an isomorphism

(1.2) $\left[BX, BY\right] = B\left[X, Y\right]$ for all $X, Y \in \zeta_0^1(V)$

Let $\bar{G}$ be the Riemannain metric tensor of M^n, we define g and g^* on V^m and $N(V)$ respectively as

(1.3) $g\left(X_1, X_2\right) = \tilde{G}\left(BX_1, BX_2\right) f,$ and

(1.4) $g^*\left(N_1, N_2\right) = \tilde{G}\left(CN_1, CN_2\right)$

For all $X_1, X_2 \in \zeta_0^1(V)$ and $N_1, N_2 \in \eta_0^1(V)$

It can be verified that g and g^* are the induced metrics on V^m and $N(V)$ respectively.

Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in M^n, then $\tilde{\nabla}$ induces a connection ∇ in $f(V)$ defined by

(1.5) $\nabla_{\bar{X}} \bar{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} \big/ f(V)$

where $\bar{X}$ and $\bar{Y}$ are arbitrary C^∞ vector fields defined along $f(V)$ and tangential to $f(V)$.

Let us suppose that M^n is a (4, 1) structure manifold with structure tensor $\tilde{\psi}$ of type (1, 1) satisfying

(1.6) $\tilde{\psi}^4 + \tilde{\psi} = 0$

Let $\tilde{L}$ and $\tilde{M}$ be the complementary distributions corresponding to the projection operators

(1.7) $\tilde{l} = -\tilde{\psi}^3, \qquad \tilde{m} = I + \tilde{\psi}^3$

where I denotes the identity operator.

From (1.6) and (1.7), we have

(1.8) **(a)** $\tilde{l} + \tilde{m} = I$ (b) $\tilde{l}^2 = \tilde{l}$ (c) $\tilde{m}^2 = \tilde{m}$

(d) $\tilde{l}\,\tilde{m} = \tilde{m}\,\tilde{l} = 0$

Let D_l and D_m be the subspaces inherited by complementary projection operators l and m respectively.
We define

$$D_l = \left\{X \in T_p(V):\ lX = X, mX = 0\right\}$$
$$D_m = \left\{X \in T_p(V):\ mX = X, lX = 0\right\}$$

Thus $T_p(V) = D_l + D_m$

Also $Ker\ l = \left\{X : lX = 0\right\} = D_m$

$Ker\ m = \left\{X : mX = 0\right\} = D_l$ at each point p of $f(V)$.

2. INVARIANT SUBMANIFOLD OF $(4,1)$ STRUCTURE MANIFOLD

We call V^m to be invariant submanifold of M^n if the tangent space $T^p\left(f(V)\right)$ of $f(V)$ is invariant by the linear mapping $\tilde{\psi}$ at each point p of $f(V)$. Thus

(2.1) $\tilde{\psi} BX = B\psi X$, for all $X \in \zeta_0^1(V)$,, and ψ being a (1,1) tensor field in V^m.

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Theorem 2.1: Let $\tilde{N}$ and N be the Nijenhuis tensors determined by $\tilde{\psi}$ and ψ in M^n and V^m respectively, then

(2.2) $\tilde{N}\left(BX, BY\right) = BN\left(X, Y\right),$ for all $X, Y \in \zeta_0^1(V)$

Proof: We have, by using (1.2) and (2.1)

(2.3)
$$\begin{aligned}\tilde{N}\left(BX, BY\right) &= \left[\tilde{\psi}BX, \tilde{\psi}BY\right] + \tilde{\psi}^2\left[BX, BY\right] - \tilde{\psi}\left[\tilde{\psi}BX, BY\right] - \tilde{\psi}\left[BX, \tilde{\psi}BY\right] \\ &= \left[B\psi X, B\psi Y\right] + \tilde{\psi}^2 B\left[X, Y\right] - \tilde{\psi}\left[B\psi X, BY\right] - \tilde{\psi}\left[BX, B\psi Y\right] \\ &= B\left[\psi X, \psi Y\right] + B\psi^2\left[X, Y\right] - \tilde{\psi}B\left[\psi X, Y\right] - \tilde{\psi}B\left[X, \psi Y\right] \\ &= B\left\{\left[\psi X, \psi Y\right] + \psi^2\left[X, Y\right] - \psi\left[\psi X, Y\right] - \psi\left[X, \psi Y\right]\right\} \\ &= BX + B\psi^3 X\end{aligned}$$

3. DISTRIBUTION $\tilde{M}$ NEVER BEING TANGENTIAL TO $f(V)$

Theorem 3.1: if the distribution $\tilde{M}$ is never tangential to $f(V)$, then

(3.1) $\tilde{m}\left(BX\right) = 0$ for all $X \in \zeta_0^1(V)$

and the induced structure ψ on V^m satisfies

(3.2) $\psi^3 = -I$

Proof: if possible $\tilde{m}\left(BX\right) \neq 0$. From (2.1) we get

(3.3) $\tilde{\psi}^3 BX = B\psi^3 X;$ from (1.7) and (3.3)

$$\begin{aligned}\tilde{m}\left(BX\right) &= \left(I + \tilde{\psi}^3\right) BX \\ &= BX + B\psi^3 X\end{aligned}$$

(3.4) $\tilde{m}\left(BX\right) = B\left[X + \psi^3 X\right]$

This relation shows that $\tilde{m}\left(BX\right)$ is tangential to $f\left(V\right)$ which contradicts the hypothesis. Thus $\tilde{m}\left(BX\right) = 0$.

Using this result in (3.4) and remembering that B is an isomorphism, we get

(3.5) $\psi^3 = -I$

Theorem 3.2: Let $\tilde{M}$ be never tangential to $f(V)$, then

(3.6) $\underset{\tilde{m}}{\tilde{N}}\left(BX, BY\right) = 0$

Proof: We have

(3.7) $\underset{\tilde{m}}{\tilde{N}}\left(BX, BY\right) = \left[\tilde{m}BX, \tilde{m}BY\right] + \tilde{m}^2\left[BX, BY\right] - \tilde{m}\left[\tilde{m}BX, BY\right] - \tilde{m}\left[BX, \tilde{m}BY\right]$

Using (1.2), (1.8) (c) and (3.1), we get (3.6).

Theorem 3.3: Let $\tilde{M}$ be never tangential to $f(V)$, then

(3.8) $\underset{\tilde{l}}{\tilde{N}}\left(BX, BY\right) = 0$

Proof: We have

(3.9) $\underset{\tilde{l}}{\tilde{N}}\left(BX, BY\right) = \left[\tilde{l}BX, \tilde{l}BY\right] + \tilde{l}^2\left[BX, BY\right] - \tilde{l}\left[\tilde{l}BX, BY\right] - \tilde{l}\left[BX, \tilde{l}BY\right]$

Using (1.2), (1.8) (a), (b) and (3.1) in (3.9); we get (3.8)

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Theorem 3.4: Let $\tilde{M}$ be never tangential to $f(V)$. Define

(3.10) $\tilde{H}(\tilde{X},\tilde{Y}) = \tilde{N}(\tilde{X},\tilde{Y}) - \tilde{N}(\tilde{m}\tilde{X},\tilde{Y}) - \tilde{N}(\tilde{X},\tilde{m}\tilde{Y}) + \tilde{N}(\tilde{m}\tilde{X},\tilde{m}\tilde{Y})$

For all $\tilde{X},\tilde{Y} \in \zeta_0^1(M)$, then

(3.11) $\tilde{H}(BX,BY) = BN(X,Y)$

Proof: Using $\tilde{X} = BX$, $\tilde{Y} = BY$ and (2.2), (3.1) in (3.10). we get (3.11).

4. DISTRIBUTION $\tilde{M}$ ALWAYS BEING TANGENTIAL TO $f(V)$

Theorem 4.1: Let $\tilde{M}$ be always tangential to $f(V)$, then

(4.1) (a) $\tilde{m}(BX) = Bm\,X$ (b) $\tilde{l}(BX) = Bl\,X$

Proof: from (3.4), We get (4.1) (a). Also

(4.2) $l = -\psi^3$

$lX = -\psi^3\,X$

(4.3) $BlX = -B\psi^3\,X$

Using (2.1) in (4.3)

(4.4) $BlX = -\tilde{\psi}^3\,BX = \tilde{l}(BX)$,

which is (4.1) (b).

Theorem 4.2: Let $\tilde{M}$ be always tangential to $f(V)$, then *l* and *m* satisfy

(4.5) (a) $l + m = I$ (b) $lm = ml = 0$ (c) $l^2 = l$ (*d*) $m^2 = m$.

Proof: Using (1.8) and (4.1) We get the results.

Theorem 4.3: If $\tilde{M}$ is always tangential to $f(V)$, then

(4.6) $\psi^4 + \psi = 0$

Proof: From (2.1)

(4.7) $\tilde{\psi}^4\,BX = B\psi^4\,X$

Using (1.6) in (4.7)

$-\tilde{\psi}\,BX = B\psi^4\,X$

$-B\psi X = B\psi^4\,X$

Or $\psi^4 + \psi = 0$ which is (4.6)

Theorem 4.4: If $\tilde{M}$ Is always tangential to $f(V)$ then as in (3.10)

(4.8) $\tilde{H}(BX,BY) = BH(X,Y)$

Proof: from (3.10) we get

(4.9) $\tilde{H}(BX,BY) = \tilde{N}(BX,BY) - \tilde{N}(\tilde{m}BX,BY) - \tilde{N}(BX,\tilde{m}BY) + \tilde{N}(\tilde{m}BX,\tilde{m}BY)$

Using (4.1) (a) and (2.2) in (4.9) we get (4.8).

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