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Group extension through r-maps

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ABSTRACT

In this paper the structure of r-map have been defined and on the basis of that group extension of G of H have been solved for which S will be a left transversal to H in G.

1.1 r −map

Def 1.1.1: Let G be a group with identity "e". A map "r" from G to G satisfying the following properties
(i) $r(e) = e$
(ii) $r^2 = r$
(iii) $r(ab) = r\big(a\, r(b)\big)$ is called r −map.

Through out this part it shall use $[r(a)]^{-1} = r(a)^{-1}$ and $r(a \cdot b) = r(ab)$, where " $\cdot$ " is group operation.

Example 1.1.2: The identity map i on the group G is a r map.

Proposition 1.1.3: Let G be a group with identity e. Let H be a subgroup of G and S be a left transversal (with identity) to H in S. Since each $a \in S$ can be uniquely written as xh where $x \in S$ and $h \in H$. Then the map $r: G \to G$ defined by $r(a) = x$ is a r −map.

Proof: $r(e) = r(ee) = e$

Now $r(a) = r\big(r(a)\big)$
$= r(x)$
$= x$
$= r(a)$

Which gives us $r^2 = r$

For the third property, let
$a = x_1h_1, b = x_2h_2$ where $x_1, x_2 \in S$ and $h_1, h_2 \in H$.

Then $r(a) = x_1$ and $r(b) = x_2$

Now $ab = x_1h_1x_2h_2$ (1)

Let $xh = x_1h_1x_2$ (2)

Therefore $r(xh) = r(x_1h_1x_2)$
$\Rightarrow x = r\big(a\, r(b)\big)$ (3)

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Using (1) and (2) it follows

$$ab = xhh_2$$

Therefore $r(ab) = r(xhh_2)$

$$\Rightarrow r(ab) = x$$

Hence $r(ab) = r\big(a\, r(b)\big)$ (by (3))

Proposition 1.1.4: Let G be the group with identity e. Let H be a subgroup of G and S be a left transversal (with identity) to H in G. Let r be a map defined in prop 1.1.3. Then the following:

(i) $r(r(a)b) = r(a)r(b) \forall a, b \in G$.

(ii) $r(x) = x \;\forall x \in S$.

(iii) $r(h) = e \forall h \in H$.

Proof: (i) Let $a = x_1h_1, b = x_2h_2$ where $x_1, x_2 \in S$ and $h_1, h_2 \in H$.

Then $r(a) = x_1$ and $r(b) = x_2$.

Now $ab = x_1h_1x_2h_2$ (1)

Let $xh = h_1x_2h_2$ (2)

Therefore

$$r(xh) = r(h_1x_2h_2)$$
$$\Rightarrow x = r(x_1^{-1}x_1h_1x_2h_2)$$
$$\Rightarrow x = r(r(a)^{-1}ab) \quad (3)$$

Then using equation (1) and (2), it follows

$$ab = x_1xh$$

Therefore $r(ab) = r(x_1xh)$

$$\Rightarrow r(ab) = x_1x$$
$$\Rightarrow r(ab) = r(a)r(r(a)^{-1}ab) \quad \text{(by (3))} \quad (4)$$

Substituting $a = r(a)$ in equation (4), it follows

$$r(r(a)b) = r\big(r(a)\big)r\left(r\big(r(a)\big)^{-1}\right)r(a)b$$
$$= r(a)r(r(a)^{-1}r(a)b)$$
$$= r(a)r(b) \quad \text{(by (ii) property of } r-\text{map)}$$

(ii) Let $x \in S$ then $x = xe$

Therefore

$$r(x) = r(xe) = x$$

(iii) Let $h \in H$ then $h = eh$. Therefore $r(h) = r(eh) = e$

Proposition 1.1.5: Let G be a group with identity e and $r: G \to G$ be a $r-$map, then $r(r(a)^{-1}a) = e$

Proof: Let $a \in G$. Then

$$a = r(a)r(a)^{-1}a$$
$$\Rightarrow r(a) = r(r(a)r(a)^{-1}a)$$
$$\Rightarrow r\big(r(a)\big) = r\big(r(r(a)r(a)^{-1}a)\big) \text{ [by (i) and (ii) properties of } r-\text{map] then,}$$
$$r(r(a)^{-1}a) = e.$$

Proposition 1.1.6: Let G be a group with identity e and $r: G \to G$ be a $r-$map. Then the set $H = \{a \in G: r(a) = e\}$ is a subgroup of G.

Proof: (i) Since $r(e) = e \Rightarrow e \in H$

So H is non-empty

(ii) Let $h_1, h_2 \in H$. Then

$$r(h_1h_2) = r(h_1 r(h_2))$$
$$= r(h_1 e)$$
$$= r(h_1)$$
$$= e$$

So it follows $h_1 h_2 \in H$

(iii) Let $h \in H$. Then

$$e = r(e)$$
$$= r(h^{-1}h)$$
$$= r(h^{-1}r(h))$$
$$= r(h^{-1}e)$$
$$= r(h^{-1})$$ (By using the properties of $r-\ map$)

So $h^{-1} \in H$

Proposition 1.1.7: Let G be a group with identity and $r : G \to G$ be a $r - map$.

Then the subset $S = \{r(a): a \in G\}$ of G is a left transversal with identity to the subgroup $H = \{a \in G: r(a) = e\}$ in G.

Proof: Suppose $S = \{r(a): a \in G\}$ is not left transversal to $H = \{a \in G: r(a) = e\}$ in G. Therefore some $a \in G$ can be written as $a = r(a_1)h_1 = r(a_2)h_2$ where $h_1, h_2 \in H$ & $h_1 \neq h_2$ and $r(a_1), r(a_2) \in S$ & $r(a_1) \neq r(a_2)$

So $r(r(a_1)h_1) = r(r(a_2)h_2)$

$\Rightarrow r(r(a_1)r(h_1) = r(r(a_2)r(h_2)$ < By (iii) property of $r - map$>

$\Rightarrow r(r(a_1)) = r(r(a_2)e)$ < Given by definitionof H >

$\Rightarrow r(r(a_1)) = r(r(a_2))$

$\Rightarrow r(a_1) = r(a_2)$ < By (ii) property of $r - map$>

And there fore $h_1 = h_2$ which is contradiction to the assumption. Thus each element $a \in G$ can be uniquely written as $r(a)h$ where $h \in H$ and $r(a) \in S$. This shows that S is the left transversal to H in G.

Also $e = r(e) \Rightarrow e \in S$.

Definition 1.1.8: A left loop is a groupoid $(S,\circ)$ with an identity element in which the equation $x \circ X = y$ possesses a unique solution for the unknown X. Groupoid $(S,\circ)$ is called a loop if the equation $Y \circ x = y$ also possesses a unique solution for the unknown Y.

Proposition 1.1.9: Let G be a group with identity e and $r : G \to G$ be a $r - map$.

Let $a_{1,} a_2 \in G$. Let S be a subset defined in proposition 8.1.7. Define a binary operation $'\circ'$ on S by

$r(a_1) \circ r(a_2)) = r(r(a_1)r(a_2))$. for all $r(a_1), r(a_2) \in S$. Then $(S,\circ)$ is a left loop.

Proof: Let $r(a_1), r(a_2) \in S$ to show that the $r(a_1) \circ X = r(a_2)$ possesses a unique solution for the unknown X in S and also contain the identity element. Now, show that $X = r(a_1)^{-1}r(a_2)$ is the unique solution

Therefore,

$$r(a_1) \circ X = r(a_2)$$
$$\Rightarrow r(r(a_1)X) = r(a_2) \text{ (by definition of } '\circ') \quad (1)$$

Put $X = r(a_1\ )^{-1}r(a_2)$ in $eq(1)$,

It follows $r(a_1) = r(a_2)$ (by (ii) property of $r - map$)

Now, let $X = r(z_1), r(z_2)$ be two solution in S, for some distinct $z_1, z_2 \in G$, of equation(1).

Then

Now $r(r(a_1)r(z_1)) = r(a_2) = r(r(a_1)r(z_2))$

$\Rightarrow r(r(a_1)z_1) = r(a_2) = r(r(a_1)z_2)$ (by (iii) property of $r - map$)

Since S is a left transversal, so two distinct elements $r(a_1)z_1, r(a_2)z_2$ in G can not be written as

$r(a_1)z_1 = r(a_2)h_1$

and $r(a_1)z_2 = r(a_2)h_2$ for distinct h_1, h_2 in H.

Therefore $z_1 = z_2$ and hence

$$r(z_1) = r(z_2)$$

Also S contains the identity element (by proposition 1.1.7)

Remark 1.1.10: Let G be a group with identity e then corresponding to a $r - map$ from $G\ to\ G, G$ can factorized as $G = SH$ where S and H are as defined above. Thus each element $a \in G$ can be uniquely written as $a = xh$ where $x \in S$ and $h \in H$.

Def 1.1.11: Let G be a group with identity e and X be a set. A map $\theta: G \times X \to X$ is called a left action of on X if

(i) $e\theta x = x$

(ii) $a_1 a_2 \theta x = a_1 \theta(a_2 \theta x)$

Proposition 1.1.12: Let G be a group with identity e and $r: G \to G$ be a $r - map$. Let S be a left transversal to H in G,

Let us define $\theta: H \times S \to S$

By $h\theta x = r(hx)$

The θ is a left action of $H\ an\ S$.

Proof: Let $x \in S$ and $h_{1,} h_2 \in H$

(i) $e\theta x = r(ex) = r(x) = x$

(ii) Now, show that

$(h_1 h_2)\theta x = h_1 \theta(h_2 \theta x)$

$L.H.S = (h_1 h_2)\theta x$

$= r((h_1 h_2)x)$

$= r(h_1 r(h_2 x))$ (by (iii) property of $r - map$)

$= h_1 \theta r(h_2 x)$

$= h_1 \theta(h_2 \theta x) = R.H.S$

1.2. NORMALITY, STABILITY AND PERFECT STABILITY OF *H*

In this section, by showing left loop $(S,\circ)$ to be a group and using some definition and results from previous propositions it will be shown that H can be a normal, stable and perfectly stable subgroup.

Proposition 1.2.1: Let G be a group with identity e and $r: G \to G$ be a $r - map$. Let S be a left transversal with identity to H in$G$. If $r - map$ satisfies the condition $r(r(a)b) = r(a)r(b)$ for all $a, b \in G$ .Then the left loop $(S,\circ)$ is a group.

Proof: Let $r(a), r(b) \in S$.

Then $r(a) \circ r(b) = r\big(r(a)r(b)\big)$

$= r(a)r\big(r(b)\big)$

$= r(a)r(b)$

Therefore $(S,\circ)$ is a group.

Now let us consider the following definition and proposition:-

It is well known that if S be a non empty set and $T(S)$ denote the set of all bijective maps for $to\ S$. Then $T(S)$ is a group with respect to the binary operation $' \cdot '$ defined by

$$(f \cdot g)(x) = g\big(f(x)\big) \forall f, g \in T(s) \text{ and } x \in S$$

This group is called the transformation group. Observe that any subgroup H of $T(S)$ acts faithfully from left on S through an action θ given by $f\theta x = f(x)$ for all $x \in S$ and $f \in H$.

So, for a left loop (S, o) define a map $f^s(zy)$ from S to S as follows consider $f^s(zy)(x)$ to be unique solution of the equation $(zoy)oX = zo\ (yox)$ where $x, y, z \in S$ and X is unknown in the equation [9] that the map. $f^s(z, y) \in T(S)$

Def 1.2.2: The subgroup of $T(S)$ generated by the subset $\{f^s(z, y): z, y \in S\}$ is called the group torsion. It is doneted by G_s.

Remark 1.2.3: $(zoy)oe = zo\,(yoe)$ implies that $f^s(z,y)(e) = e$ for all $z,y \in S$. Thus G_s is the subgroup of $T(S - \{e\})$ also.

Proposition 1.2.4: A left loop (S,o) is a group if and only if its group torsion G_s is trivial

Proof: $G_s = \{I_s\}$ if and only if $f^s(z,y) = I_s$ for all $z,y \in S$, that is $(zoy)ox = zo(yoz)\ \forall x,y,z \in S$. Thus the result follows by observing that a left loop S is a group if and only if the binary operation $'o'$ of the left loop is associative.

Corollary 1.2.5: Let G be a group with identity e and $r: G \to G$ be a $r - map$ satisfying the condition
$r(r(a_1)a_2) = r(a_1)r(a_2)$ for all $a_1, a_2 \in G$

Then group torsion G_S of every left loop determined by every left transversal S of subgroup H in G is trivial.

Proof: Proof follows from the proposition$(1.2.1), (1.2.4)$.

Proposition 1.2.6: A subgroup H of a group G is normal if and only if the group trosion of every left transversal of H in G is trivial.

Proof: Proof follows from the Corollary$(1.2.5)$.

Corollary 1.2.7: Let G be a group with identity e and $r: G \to G$ be a $r - map$ satisfying the condition
$r(r(a_1)a_2) = r(a_1)r(a_2)$

For all $a_1, a_2 \in G$, then the subgroup
$H = \{a \in G: r(g) = e\}$ of group G is normal.

Proof: Proof follows from Corollary $(1.2.5)$ and proposition$(1.2.6)$.

Definition 1.2.8: A subgroup H of a group G is called stable if group. Torsions of all left transversals to H in G.

Definition 1.2.9: A subgroup H of a group G is called perfectly stable if all left transversals to H in G are isomorphic (as left loop)

Remark 1.2.10: If H be a normal subgroup of a group G then group Torsions of all left transversals to H in G are trivial (by proposition 1.2.6). So H is stable. And also if H be a normal subgroup of a group G then all left transversals to H in G are isomorphic (as left loop) to the quotient group G/H. Then H is group with identity $'e'$ and $r: G \to G$ be a $r - map$ satisfying the condition $r(r(a_1)a_2) = r(a_1)r(a_2)$

For all $a_1, a_2 \in G$, then the subgroup $H = \{a \in G: r(a) = e\}$ of group G is both stable and perfectly stable (by Corollary $(1.2.7)$.

Proposition 1.2.11: Let G be a group with identity $'e'$. Then the total number of distinct $r - map$ on G is the total number of distinct factorizations of G as SH where H is a subgroup of G then and S is a left transversals (with identity) to H in G.

Proof: Since every G can be written as $G = SH$ where H is a subgroup of G and S is a left transversals to H in G. $r(a) = r(xh) = x$ as a $r - map$ (proposition 1.1.3). But if defined any other map except this then it cannot satisfy the condition $r^2 = r$. Therefore it is not a $r - map$ hence the result.

1.3. EXTENTION OF *H*

Let G be a group with identity $'e$ and $r: G \in G$ be a $r - map$. Let H & S be as defined in proposition 1.1.6 and 1.1.7 respectively. Let $x,y \in S$ and $h \in H$ then it can easily observed that $x \cdot y = r(xy)f(x,y)$ and $h \cdot x = r(hx)\sigma_x(h)$ for some $f(x,y), \sigma_x(h)$ and $r(xy), r(hx) \in S$ and, f is a map for $S \times S$ to H and for a fixed $x \in S, \sigma_x$ is a map from $H\ to\ H$.

Theorem 1.3.1: Let G be a group with identity $'e'$ and $r: G \to G$ be a $r - map$. Let H & S be as defined in proposition 1.1.6 and 1.1.7 respectively. Let H act on S from left through an action θ as defined in proposition 1.1.12. Then the following:

(i) $\sigma_x(h_1h_2) = \sigma_{r(h_2x)}(h_1)\sigma_x(h_2)$
(ii) $r\big(x(yz)\big) = r(r(xy)f(x,y)z)$
(iii) $r(h(xy)) = r(r(hx)\sigma_x(h)y)$

(iv) $f(x, r(yz))f(y, z) = f(r(xy), r(f(x, y)z))\sigma_z(f(x, y))$
(v) $\sigma_{r(xy)}(h)f(x, y) = f(r(hx), r(\sigma_x(h)y))\sigma_y(\sigma_x(h))$, where $x, y, z \in S$ and $h, h_1, h_2, \in H$.

Proof:
(i) Let $x \in S$ and $h_1, h_2, \in H$

Then using $h \cdot x = r(hx)\sigma_x(h)$ and associativity of G it follows

$$\begin{aligned} r\big((h_1h_2)x\big)\sigma_x(h_1h_2) &= (h_1.h_2) \cdot x \\ &= h_1 \cdot (h_2 \cdot x) \\ &= h_1\big(r(h_2x)\sigma_x(h_2)\big) \\ &= \big(h_1 r(h_2x)\big)\sigma_x(h_2) \\ &= r\big(h_1 r(h_2x)\big)\sigma_{r(h_2x)}(h_1)\sigma_x(h_2) \\ &= r\big(h_1(h_2x)\big)\sigma_{r(h_2x)}(h_1)\sigma_x(h_2) \end{aligned}$$

So it follows that
$r\big((h_1h_2)x\big) = r\big(h_1(h_2x)\big) \Rightarrow H$ acts on S from left and $\sigma_x(h_1h_2) = \sigma_{r(h_2x)}(h_1)\sigma_x(h_2)$

(ii) and (iii)

Let $x, y, z \in S$. Then using $x \cdot y = r(xy)f(x, y)$ and associativity of G, it follows

$$\begin{aligned} r\big(x(yz)\big)f\big(x, r(yz)\big)f(y, z) &= r\big(xr(yz)\big)f\big(x, r(yz)\big)f(y, z) \\ &= \big(x \cdot r(yz)\big)f(y, z) \\ &= x \cdot \big(r(yz)f(y, z)\big) \\ &= x \cdot (y \cdot z) \\ &= (x \cdot y) \cdot z \\ &= \big(r(xy)f(x, y)\big)z \\ &= r(xy)(f(x, y)z) \\ &= r(xy)\Big(r(f(x, y)z)\sigma_z\big(f(x, y)\big)\Big) \\ &= \big(r(xy)r(f(x, y)z)\big)\sigma_z\big(f(x, y)\big) \\ &= r\big(r(xy)r(f(x, y)z)\big)f\big(r(xy), r(f(x, y)z)\big)\sigma_z\big(f(x, y)\big) \\ &= r\big(r(r(xy)f(x, y)z)\big)f\big(r(xy), r(f(x, y)z)\big)\sigma_z\big(f(x, y)\big) \\ &= r(r(xy)f(x, y)z)f\big(r(xy), r(f(x, y)z)\big)\sigma_z\big(f(x, y)\big) \end{aligned}$$

Thus $r\big(x(yz)\big) = r(r(xy)f(x, y)z)$

and $f\big(x, r(yz)\big)f(y, z) = f\big(r(xy), r(f(x, y)z)\big)\sigma_z\big(f(x, y)\big)$

(iii) and (iv)

Let $x, y \in S$ and $h \in H$ then similarly using $\cdot\, y = r(xy)f(x, y)$, $h \cdot x = r(hx)\sigma_x(h)$ and associativity of G, it follows

$$r(h(xy)) = r(r(hx)\sigma_x(h)y)$$

and

$$\sigma_{r(xy)}(h)f(x, y) = f(r(hx), r(\sigma_x(h)y))\sigma_y(\sigma_x(h))$$

Proposition 1.3.2: Let G be a group with identity "e" and $r: G \to G$ be a $r - map$. Let H and S be as defined in proposition 1.1.6 and 1.1.7 respectively.

Let $x \in S$. Then
(i) $\sigma_x(e) = e$
(ii) $\sigma_e = I_H$,where I_H is identity map on H
(iii) $f(x, e) = e = f(e, x)$

Proof:
(i)

$$\begin{aligned} \sigma_x(e) &= \sigma_x(e \cdot e) \\ &= \sigma_{r(ex)}(e)\sigma_x(e) \\ &= \sigma_{r(x)}(e)\sigma_x(e) \\ &\Rightarrow \sigma_x(e) = e \end{aligned}$$

(ii) Let $h \in H$.

Then $he = r(he)\sigma_e(h)$
$\Rightarrow h = e\sigma_e(h)$
$\Rightarrow h = \sigma_e(h)$
$\Rightarrow \sigma_e = I_H$

(iii) $ex = r(ex)f(e,x)$
$\Rightarrow x = r(x)f(e,x)$
$\Rightarrow xe = xf(e,x)$
$\Rightarrow e = f(e,x)$

Similarly, it can be easily observed that $f(x,e) = e$.

Theorem 1.3.4: Let G be a group with identity e and $r: G \to G$ be a $r - map$. Let H and S be as defined in proposition 1.1.6 and 1.1.7 respectively. Then G be an extension of the subgroup H with a left transversal S to H in G.

Proof: Let $G = SH$ denoted the Cartesian product of SH. Let us denoted an ordered pair (x,h) by xh.

$$xa \cdot yb = r\big(x(ay)\big)f\big(x, r(ay)\big)\sigma_y(a)b \qquad (1)$$

Associativity of the binary operation " $\cdot$ " is as follows:-

Let $h,k,l \in H$ and $x,y,z \in S$ then

$$\begin{aligned}
(xa \cdot yb) \cdot zc
&= \big[r\big(x(ay)\big)f\big(x,r(ay)\big)\sigma_y(a)b\big] \cdot zc \\
&= r\Big(r\big(x(ay)\big)r\big(f\big(x,r(ay)\big)\sigma_y(a)bz\big)\Big) f\Big(r\big(x(ay)\big), r\big(f\big(x,r(ay)\big)\sigma_y(a)bz\big)\Big)\sigma_z\big(f\big(x,r(ay)\big)\sigma_y(a)b\big)c \\
&= r\Big(r\big(x(ay)\big)r\big(f\big(x,r(ay)\big)\sigma_y(a)bz\big)\Big) f\Big(r\big(x(ay)\big), r\big(f\big(x,r(ay)\big)\sigma_y(a)bz\big)\Big) \\
&\qquad \sigma_{r\left(\sigma_y(a)r(b\theta z)\right)}\Big(f\big(x,r(ay)\big)\Big)\sigma_{r(bz)}\Big(\sigma_y(a)\Big)\sigma_z(b)c \\
&= r\Big(r\big(x(ay)\big)\Big)r\big(r(\sigma_y(a)b)z\big)f\Bigg(x, r\bigg(a\Big(y\big(r(bz)\big)\Big)\bigg)\Bigg) f\Big(r(ay), r\Big(\sigma_y(a)r(bz)\Big)\Big)\sigma_{r(bz)}\Big(\sigma_y(a)\Big)\sigma_z(b)c \\
&= r\bigg(x\Big(a\big(y(bz)\big)\Big)\bigg) f\bigg(x, r\Big(a\big(y(bz)\big)\Big)\bigg)\sigma_{r(yr(bz))}(a)f\big(y,r(bz)\big)\sigma_z(b)c \\
&= xa \cdot \big[r\big(y(bz)\big)f\big(y,r(bz)\big)\sigma_z(b)c\big] \\
&= xa \cdot (yb \cdot zc)
\end{aligned}$$

ee is the identity element of $G = SH$

Inverse of xa is $r(a^{-1}x')\sigma_{x'}(a^{-1})\Big(f(x,x')\Big)^{-1}$.

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