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# POSITIVE IMPLICATIVE BIPOLAR VAGUE IDEALS IN BCK-ALGEBRAS

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***ABSTRACT***

*The focus of this paper is to investigate the notion of bipolar vague ideals in BCK-algebra. Further we discuss the relation between bipolar vague ideal and a positive implicative bipolar vague ideal.*

***Keywords:*** *bipolar Vague ideal, bipolar vague subalgebra, positive implicative bipolar vague ideal, implicative bipolar vague ideal.*

## 1. INTRODUCTION

In the traditional fuzzy sets, the membership degree of elements range over the interval [0,1]. Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irreverent elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy sets representation. Based on these observations, Lee [9] introduced an extension of fuzzy sets named bipolar- valued fuzzy sets. He applied the notion of bipolar- valued fuzzy sets to BCK/ BCH-algebras [9] and introduced the concept of bipolar fuzzy subalgebra/ideals of a BCK/ BCH-algebra and investigated several properties. Interval-valued fuzzy sets were first introduced by Zadeh [10] as a generalization of fuzzy sets. This idea gives the simplest method to capture the imprecision of the membership grade for fuzzy sets. The concept of vague set [1] is introduced in 1993 by W.L. Gau and Buehrer. D. J. In a vague set A, there are two membership functions: a truth membership function $t_A : U \to [0,1]$ and a false membership function $f_A : U \to [0,1]$, where $t_A(u)$ is a lower bound on the grade of membership of $u$ derived from the "evidence for $u$". $f_A(u)$ is a lower bound on the negation of $u$ derived from the "evidence against $u$" and $t_A(u) + f_A(u) \le 1$. Thus the grade of membership of $u$ in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of [0, 1]. This indicates that if the actual grade of membership of $u$ is $\mu(u)$, then $t_A(u) \le \mu(u) \le 1 - f_A(u)$. Vague set is an extension of fuzzy sets. The idea of vague sets is that the membership of every elements which can be divided into two aspects including supporting and opposing. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1].

In this paper, we apply the notion of a bipolar vague set to BCK/BCI-algebras. We introduce the concept of bipolar vague sub-algebras/ideals of BCK algebra, positive implicative bipolar vague ideal and analyse their properties.

## 2. PRELIMINARIES

**Definition 2.1 [11]:** An algebra (X, $*$, 0) of type (2, 0) is called a BCK-algebra if it satisfies the axioms:

(a1) $((x*y)*(x*z))*(z*y) = 0$,

(a2) $(x*(x*y))*y = 0$,

(a3) $x*x = 0$,

(a4) $x*y = 0$ and $y*x = 0$ imply that $x = y$

(a5) $0*x = 0$ for all $x, y, z \in X$.

**Definition 2.2 [11]:** A partial ordering "$\le$" on X can be defined by $x \le y$ if and only if $x*y = 0$.

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**Definition 2.3 [11]:** In any BCK-algebra X the following holds:
(P1) $x*0 = x$
(P2) $x*y \le x$
(P3) $(x*y)*z = (x*z)*y$
(P4) $(x*z)*(y*z) \le x*y$
(P5) $x*(x*(x*y)) = x*y$
(P6) $x \le y \Rightarrow x*z \le y*z$ and $z*y \le z*x$, for all $x, y, z \in X$ .

**Definition 2.4 [7]:** A subset S of a BCK-algebra X is called a subalgebra of X if $x*y \in S$ whenever $x, y \in S$ .

**Definition 2.5 [11]:** A non empty subset I of a BCK-algebra X is called an ideal of X if it satisfies:
(C1) $0 \in I$,
(C2) $x*y \in I$ and $y \in I$ imply $x \in I$

**Proposition 2.6 [10]:** In a BCK-algebra X, the following holds, for all $x, y, z \in X$ .
(i) $((x*z)*z)*(y*z) \le (x*y)*z.$
(ii) $(x*z)*(x*(x*z)) = (x*z)*z$
(iii) $(x*(y*(y*x)))*(y*(x*(y*(y*x)))) \le x*y$ .

**Definition 2.6 [6]:** Let X be the universe of discourse. A bipolar- valued fuzzy set A in X is an object having the form A=$\{(x, t_A^+(x), t_A^-(x)) / x \in X\}$ where $t_A^+(x): X \to [0,1]$ and $t_A^-(x): X \to [-1,0]$ are mappings. The positive membership degree $t_A^+(x)$ denotes the satisfaction degree of an element $x$ to the property corresponding to a bipolar-valued fuzzy sets $A = \{(x, t_A^+(x), t_A^-(x)) / x \in X\}$, and the negative membership degree $t_A^-(x)$ denote the satisfaction degree of $x$ to some implicit counter property of $A = \{(x, t_A^+(x), t_A^-(x)) / x \in X\}$

## 3. BIPOLAR VAGUE SET

**Definition 3.1:** Let X be the universe of discourse. A bipolar-valued vague set A in X is an object having the form A ={(x, [$t^+{}_A$(x),$1 - f^+{}_A$(x)], [$t^-{}_A$(x), $1 - f^-{}_A$(x)]) /x∈X} where [ $t^+{}_A$, $1 - f^+{}_A$]: X→[0, 1] and $t^-{}_A$, $1 - f^-{}_A$:X →[-1,0] are mappings and $t^+{}_A$+$1 - f^+{}_A$ ≤ 1, -1 ≤ $t^-{}_A$+$1 - f^-{}_A$. The positive membership degree [$t^+{}_A$(x),$1 - f^+{}_A$(x)] denotes the satisfaction region of an element x to the property corresponding to a bipolar-valued vague set A ={(x, [$t^+{}_A$(x),$1 - f^+{}_A$(x)], [ $t^-{}_A$(x), $1 - f^-{}_A$(x)]) / x∈X}, and the negative membership degree [$t^-{}_A$(x), $1 - f^-{}_A$(x)] denote the satisfaction region of x to some implicit counter-property of
A = {(x, [$t^+{}_A$(x),$1 - f^+{}_A$(x)], [ $t^-{}_A$(x), $1 - f^-{}_A$(x)]) / x∈X}.

**Remark 3.2:** If $t_A^+(x) \neq 0$ and $1 - f_A^+(x) \neq 0$, $t_A^-(x)$ and $-1 - f_A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $A = \{(x, [t_A^+(x), 1 - f_A^+(x)], [t_A^-(x), -1 - f_A^-(x)]) / x \in X\}$. If $t_A^+(x) = 0$ and $1 - f_A^+(x) = 0$, $t_A^-(x) \neq 0$ and $-1 - f_A^-(x) \neq 0$, it is the situation that x does not satisfy the property of $A = \{(x, [t_A^+(x), 1 - f_A^+(x)], [t_A^-(x), -1 - f_A^-(x)]) / x \in X\}$ but somewhat satisfies the counter-property of $A = \{(x, [t_A^+(x), 1 - f_A^+(x)], [t_A^-(x), -1 - f_A^-(x)]) / x \in X\}$. It is possible for an element x to be $t_A^+(x) \neq 0$, $1 - f_A^+(x) \neq 0$ and $t_A^-(x) \neq 0$, $-1 - f_A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain.

**Note 3.3:** For the sake of simplicity, we shall use the symbol A = (X, $V^+{}_A$, $V^-{}_A$) for the bipolar-valued vague set A ={(x, [$t^+{}_A$(x),$1 - f^+{}_A$(x)], [ $t^-{}_A$(x), $1 - f^-{}_A$(x)]) / x∈X}, and use the notion of bipolar vague sets instead of the notion of bipolar-valued vague sets, where $V^+{}_A$ = [$t^+{}_A$,$1 - f^+{}_A$] and $V^-{}_A$ = [$t^-{}_A$, $1 - f^-{}_A$].

## 4. BIPOLAR VAGUE SUB-ALGEBRA

**Definition 4.1:** A bipolar vague set $A = (X; v_A^+, v_A^-)$ in X is called a bipolar vague sub-algebra of X if it satisfies:

$$v_A^+(x*y) \ge \min\{v_A^+(x), v_A^+(y)\}$$

 

$v_A^-(x*y) \le \max\{v_A^-(x), v_A^-(y)\}$ for all $x, y \in X$, that is

$t_A^+(x*y) \ge \min\{t_A^+(x), t_A^+(y)\}$

$1 - f_A^+(x*y) \ge \min\left\{1 - f_A^+(x), 1 - f_A^+(y)\right\}$

$t_A^-(x*y) \le \max\{t_A^-(x), t_A^-(y)\}$

$1 - f_A^-(x*y) \le \max\left\{1 - f_A^-(x), 1 - f_A^-(y)\right\}$.

**Example 4.2:** Consider a BCK- algebra X= {0, a, b, c} with the following Cayley table:

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | a |
| b | b | a | 0 | b |
| c | c | c | c | 0 |

Let $A = (X; v_A^+, v_A^-)$ be a bipolar vague set in X defined by,

| | 0 | a | b | c |
|---|---|---|---|---|
| $v_A^+$ | [0.6, 0.8] | [0.6, 0.8] | [0.3, 0.4] | [0.6, 0.8] |
| $v_A^-$ | [-0.7, -0.5] | [-0.7, -0.5] | [-0.2, -0.1] | [-0.7, -0.5] |

Then $A = (X; v_A^+, v_A^-)$ is a bipolar vague sub-algebra of X.

**Proposition 4.3:** If $A = (X; v_A^+, v_A^-)$ is a bipolar vague sub-algebra of X, then $v_A^+(0) \ge v_A^+(x)$ and $v_A^-(0) \le v_A^-(x)$ for all $x \in X$ .

**Proof:** Let $x \in X$ . Then $v_A^+(0) = v_A^+(x*x) \ge \min\{v_A^+(x), v_A^+(x)\} = v_A^+(x)$ and

$$v_A^-(0) = v_A^-(x*x) \le \min\{v_A^-(x), v_A^-(x)\} = v_A^-(x).$$

This completes the proof.

**Definition 4.4:** A bipolar vague set $A = (X; v_A^+, v_A^-)$ of a BCK algebra X is called a bipolar vague ideal of X if the following conditions are true:

(1) $v_A^+(0) \ge v_A^+(x)$ and $v_A^-(0) \le v_A^-(x)$

(2) $v_A^+(x) \ge \min\{v_A^+(x*y), v_A^+(y)\}$ and $v_A^-(x) \le \max\{v_A^-(x*y), v_A^-(y)\}$

That is $t_A^+(0) \ge t_A^+(x)$, $1 - f_A^+(0) \ge 1 - f_A^+(x)$ and $t_A^-(0) \le t_A^-(x)$, $1 - f_A^-(0) \le 1 - f_A^-(x)$

$t_A^+(x) \ge \min\{t_A^+(x*y), t_A^+(y)\}$, $1 - f_A^+(x) \ge \min\{1 - f_A^+(x*y), 1 - f_A^+(y)\}$

$t_A^-(x) \le \max\{t_A^-(x*y), t_A^-(y)\}$, $1 - f_A^-(x) \le \max\{1 - f_A^-(x*y), 1 - f_A^-(y)\}$

**Proposition 4.5:** Let $A = (X; v_A^+, v_A^-)$ be a bipolar vague ideal of X. If the inequality $x*y \le z$ holds in X, then

$v_A^+(x) \ge \min\{v_A^+(y), v_A^+(z)\}$

$v_A^-(x) \le \max\{v_A^-(y), v_A^-(z)\}$

**Proof:** Let $x, y, z \in X$ be such that $x*y \le z$ . Then $(x*y)*z = 0$ , and so

$$\begin{aligned} v_A^+(x) &\ge \min\{v_A^+(x*y), v_A^+(y)\} \\ &\ge \min\{\min\{v_A^+((x*y)*z), v_A^+(z)\}, v_A^+(y)\} \\ &= \min\{\min\{v_A^+(0), v_A^+(z)\}, v_A^+(y)\} \\ &= \min\{v_A^+(y), v_A^+(z)\} \end{aligned}$$

And $v_A^-(x) \le \max\{v_A^-(x*y), v_A^-(y)\}$
$\le \max\{\max\{v_A^-((x*y)*z), v_A^-(z)\}, v_A^-(y)$
$= \max\{\max\{v_A^-(0), v_A^-(z)\}, v_A^-(y)\}$
$= \max\{v_A^-(y), v_A^-(z)\}$

Hence the proof.

**Proposition 4.6:** Let $A = (X; v_A^+, v_A^-)$ be a bipolar vague ideal in X. If the inequality $x \le y$ holds in X, then $v_A^+(x) \ge v_A^+(y)$ and $v_A^-(x) \le v_A^-(y)$.

**Proof:** Let $x, y \in X$ be such that $x \le y$. Then

$v_A^+(x) \ge \min\{v_A^+(x*y), v_A^+(y)\} = \min\{v_A^+(0), v_A^+(y)\} = v_A^+(y)$ and

$v_A^-(x) \le \min\{v_A^-(x*y), v_A^-(y)\} = \min\{v_A^-(0), v_A^-(y)\} = v_A^-(y)$

This completes the proof.

**Theorem 4.7:** If A and B are bipolar vague ideals of X, so is their intersection.

**Proof:** Let $C = A \cap B$, then $v_C^+(0) = \min\{v_A^+(0), v_B^+(0)\} \ge \min\{v_A^+(x), v_B^+(x)\} = v_C^+(x)$ for any $x \in X$. Furthermore, for any $x, y \in X$, we have

$v_C^+(x) = \min \{v_A^+(x), v_B^+(x)\} \ge \min\{\min\{v_A^+(x*y), v_A^+(y)\}, \min\{v_B^+(x*y), v_B^+(y)\}\}$
$= \min\{v_C^+(x*y), v_C^+(y)\}$ *and* $v_C^+(0) = \max\{v_A^-(0), v_A^-(0)\} \le \max \{v_A^-(x), v_B^-(x)\} = v_C^-(x)$
*for any* $x \in X$

For any $x, y \in X$, we have

$v_C^-(x) = \max \{v_A^-(x), v_B^-(x)\}$
$\le \max\{\max\{v_A^-(x*y), v_A^-(y)\}, \max\{v_B^-(x*y), v_B^-(y)\}\} = \max\{v_C^-(x*y), v_C^-(y)\}$

**Theorem 4.8:** In a BCK- algebra X, every bipolar vague ideal of X is a bipolar vague sub-algebra of X.

**Proof:** Let $A = (X; v_A^+, v_A^-)$ be a bipolar vague ideal of a BCK- algebra X. Since $x*y \le x$ for all $x, y \in X$, it follows from proposition 4.6 that $v_A^+(x*y) \ge v_A^+(x)$ and $v_A^-(x*y) \le v_A^-(x)$, so that

$v_A^+(x*y) \ge v_A^+(x) \ge \min\{v_A^+(x*y), v_A^+(y)\} \ge \min\{v_A^+(x), v_A^+(y)\}$ and

$v_A^-(x*y) \le v_A^-(x) \le \max\{v_A^-(x*y), v_A^-(y)\} \le \max\{v_A^-(x), v_A^-(y)\}$.

Hence $A = (X; v_A^+, v_A^-)$ is a bipolar vague sub-algebra of X.

The converse of the above is not true in general.

**Example 4.9:** In example 4.2the bipolar vague sub-algebra $A = (X; v_A^+, v_A^-)$ is not a bipolar vague ideal of X. Since $t_A^+(b) = 0.3 \ge 0.6 = \min\{t_A^+(b*a), t_A^+(a)\}$, $1 - f_A^+(b) = 0.4 \ge 0.8 = \min\{1 - f_A^+(b*a), 1 - f_A^+(a)\}$

**Corollary 4.10:** For a bipolar Vague set $A = (X; v_A^+, v_A^-)$ in a BCK- algebra X, the following condition hold:

$v_A^+((x*z)*(y*z)) \ge v_A^+((x*y)*z)$ and

$v_A^+((x*z)*(y*z)) \ge v_A^+((x*y)*z)$

## 5. POSITIVE IMPLICATIVE BIPOLAR VAGUE IDEALS

**Definition 5.1:** A bipolar vague $A = (X; v_A^+, v_A^-)$ of BCK- algebra X is called a positive implicative bipolar vague ideal of X if it satisfies:

(1) $v_A^+(0) \geq v_A^+(x)$ and $v_A^-(0) \leq v_A^-(x)$

(2) $v_A^+(x*z) \geq \min\{v_A^+((x*y)*z), v_A^+(y*z)\}$ and

$v_A^-(x*z) \leq \max\{v_A^-((x*y)*z), v_A^-(y*z)\}$

**Definition 5.2:** A bipolar vague $A = (X; v_A^+, v_A^-)$ of BCK- algebra X is called a implicative bipolar vague ideal of X if it satisfies:

(1) $v_A^+(0) \geq v_A^+(x)$ and $v_A^-(0) \leq v_A^-(x)$

(2) $v_A^+(x) \geq \min\{v_A^+((x*(y*x))*z), v_A^+(z)\}$ and

$v_A^-(x) \leq \max\{v_A^-((x*(y*x))*z), v_A^-(z)\}$

**Theorem 5.3:** Every positive implicative bipolar vague ideal is a bipolar vague ideal.

**Proof:** Let $A = (X; v_A^+, v_A^-)$ be a positive implicative bipolar vague ideal of X. If we take z=0 in Definition 5.1(2) and use (P1), then we get $v_A^+(x) \geq \min\{v_A^+(x*y), v_A^+(y)\}$ and $v_A^-(x) \leq \max\{v_A^-(x*y), v_A^-(y)\}$. Hence $A = (X; v_A^+, v_A^-)$ is a bipolar vague ideal of X.

The following example shows that the converse of the above may not be true.

**Example 5.4:** Consider a BCK- algebra X= {0, a, b, c, d} with the following Cayley table:

| * | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | 0 | 0 |
| b | b | b | 0 | 0 | 0 |
| c | c | c | c | 0 | 0 |
| d | d | c | d | a | 0 |

Let $A = (X; v_A^+, v_A^-)$ be a bipolar vague set in X defined by,

| | 0 | a | b | c | d |
|---|---|---|---|---|---|
| $v_A^+$ | [0.7, 0.8] | [0.2, 0.4] | [0.7, 0.8] | [0.2, 0.4] | [0.2, 0.4] |
| $v_A^-$ | [-0.8, 0.6] | [-0.7, 0.5] | [-0.8, 0.6] | [-0.7, 0.5] | [-0.7,-0.5] |

It is routine to verify that $A = (X; v_A^+, v_A^-)$ is a vague ideal of X. But it is not a positive implicative vague ideal of X. Since $v_A^+(b*c) \ngeq \min\{v_A^+((b*c)*c), v_A^+(c*c)\}$..

**Theorem 5.5:** For a bipolar vague ideal $A = (X; v_A^+, v_A^-)$ of X, the following conditions are equivalent:

(1) A is a positive implicative bipolar vague ideal of X.

(2) A satisfies $v_A^+(x*y) \geq v_A^+((x*y)*y)$ *and* $v_A^-(x*y) \leq v_A^-((x*y)*y)$

**Proof:** Assume that A is a positive implicative bipolar vague ideal of X. If we put z=y in definition 5.1(2) and use (a3) and definition 4.4(2), then

$v_A^+(x*y) \geq \min\{v_A^+((x*y)*y), v_A^+(y*y)\} = \min\{v_A^+((x*y)*y), v_A^+(0)\} = v_A^+((x*y)*y)$ and

$v_A^-(x*y) \leq \max\{v_A^-((x*y)*y), v_A^-(y*y)\} = \max\{v_A^-((x*y)*y), v_A^-(0)\} = v_A^-((x*y)*y)$ for all $x, y \in X$ .

Conversely, Suppose that A satisfies $v_A^+(x*y) \geq v_A^+((x*y)*y)$ *and* $v_A^-(x*y) \leq v_A^-((x*y)*y)$. Note that $((x*z)*z)*(y*z) \leq (x*y)*z$ for all $x, y, z \in X$ . By using (2) and proposition (2.6) and definition 4.4(2), we have

$v_A^+(x*z) \geq v_A^+((x*z)*z) \geq \min\{v_A^+(((x*z)*z)*(y*z)), v_A^+(y*z)\}$

$\geq \min\{v_A^+((x*y)*z),\ v_A^+(y*z)\}$ and

$$v_A^-(x*z) \le v_A^-((x*z)*z) \le \max\{v_A^-(((x*z)*z)*(y*z)), v_A^-(y*z)\}$$
$$\le \max\{v_A^-((x*y)*z),\ v_A^-(y*z)\}.$$

**Theorem 5.6:** For a bipolar vague set $A=(X;v_A^+,v_A^-)$ of X, the following conditions are equivalent:

(1) A is a positive implicative bipolar vague ideal of X.

(2) A satisfies $v_A^+(0) \ge v_A^+(x)$ , $v_A^-(0) \le v_A^-(x)$ and

$$v_A^+(x*y) \ge \min\{v_A^+(((x*y)*y)*z),\ v_A^+(z)\}$$
$$v_A^-(x*y) \le \max\{v_A^-(((x*y)*y)*z),\ v_A^-(z)\}.$$

**Proof:** Assume that A is a positive implicative bipolar vague ideal of X. Then A is a bipolar vague ideal of X by Theorem 5.3 and so we have

$$v_A^+(x*y) \ge \min\{v_A^+((x*y)*z), v_A^+(z)\} = \min\{v_A^+((x*z)*y), v_A^+(z)\}$$
$$= \min\{v_A^+(((x*z)*y)*(y*y)), v_A^+(z)\} \ge \min\{v_A^+(((x*z)*y)*y), v_A^+(z)\}$$
$$= \min\{v_A^+(((x*y)*y)*z), v_A^+(z)\} \text{ and}$$

$$v_A^-(x*y) \le \max\{v_A^-((x*y)*z), v_A^-(z)\} = \max\{v_A^-((x*z)*y), v_A^-(z)\}$$
$$= \max\{v_A^-(((x*z)*y)*(y*y)), v_A^-(z)\} \le \max\{v_A^-(((x*z)*y)*y), v_A^-(z)\}$$
$$= \max\{v_A^-(((x*y)*y)*z), v_A^-(z)\}.$$

Conversely, let A satisfies (2), we have

$v_A^+(x) = v_A^+(x*0) \ge \min\{v_A^+(((x*0)*0)*y), v_A^+(y)\} = \min\{v_A^+(x*y), v_A^+(y)\}$ and

$v_A^-(x) = v_A^-(x*0) \le \max\{v_A^-(((x*0)*0)*y), v_A^-(y)\} = \max\{v_A^-(x*y), v_A^-(y)\}$

Hence A is a bipolar vague ideal of X. If we put z=0 in (2), then

$v_A^+(x*y) \ge \min\{v_A^+(((x*y)*y)*0), v_A^+(0)\} = v_A^+((x*y)*y)$ and

$v_A^-(x*y) \le \max\{v_A^-(((x*y)*y)*0), v_A^-(0)\} = v_A^-((x*y)*y)$ for all $x, y \in X$ it follows from the Theorem 5.5 that $A=(X;v_A^+,v_A^-)$ is a positive implicative bipolar vague ideal of X

**Theorem 5.7:** Every implicative bipolar vague ideal of a BCK- algebra X is a positive implicative bipolar vague ideal of X.

**Proof:** Let $A=(X;v_A^+,v_A^-)$ be an implicative bipolar vague ideal of a BCK-algebra X. Since

$$((x*z)*z)*(y*z) \le (x*z)*y = (x*y)*z$$

Using proposition 4.6 we obtain

$v_A^+(((x*z)*z)*(y*z)) \ge v_A^+((x*y)*z)$ and $v_A^-(((x*z)*z)*(y*z)) \le v_A^-((x*y)*z)$ also that $(x*z)*(x*(x*z)) = (x*(x*(x*z)))*z) = (x*z)*z$ it follows from Definition 5.2(2)

$$v_A^+(x*z) \ge \min\{v_A^+(((x*z)*(x*(x*z)))*(y*z)), v_A^+(y*z)\}$$
$$= \min\{v_A^+(((x*z)*z)*(y*z)), v_A^+(y*z)\} \ge \min\{v_A^+((x*y)*z), v_A^+(y*z)\} \quad a \quad n$$

$$v_A^-(x*z) \le \max\{v_A^-(((x*z)*(x*(x*z)))*(y*z)), v_A^-(y*z)\}$$
$$= \max\{v_A^-(((x*z)*z)*(y*z)), v_A^-(y*z)\} \le \max\{v_A^-((x*y)*z), v_A^-(y*z)\}$$

Hence $A=(X;v_A^+,v_A^-)$ is a positive implicative bipolar vague ideal of X.

**Theorem 5.8:** If $A=(X;v_A^+,v_A^-)$ is a positive implicative bipolar vague ideal of X, then for any

$(i)\ ((x*y)*y)*a \le b \ \Rightarrow v_A^+(x*y) \ge \min\{v_A^+(a), v_A^+(b)\}$ and $v_A^-(x*y) \le \max\{v_A^-(a), v_A^-(b)\}$

$(ii)\ ((x*y)*z)*a \le b \ \Rightarrow v_A^+((x*z)*(y*z)) \ge \min\{v_A^+(a), v_A^+(b)\}$ and

$v_A^-((x*z)*(y*z) \le \max\{v_A^-(a), v_A^-(b)\}$

**Proof:** Suppose $A=(X;v_A^+,v_A^-)$ is a positive implicative bipolar vague ideal of X

(i) Let $x,y,z\in X$ such that $((x*y)*y)*a\le b$, we have $v_A^+((x*y)*y)\ge \min\{v_A^+(a),v_A^+(b)\}$ and it follows that

$$v_A^+(x*y)\ge \min\{v_A^+((x*y)*y),v_A^+(y*y)\}=\min\{v_A^+((x*y)*y),v_A^+(0)\}$$

$$=v_A^+((x*y)*y)\ge \min\{v_A^+(a),v_A^+(b)\} \text{ and } v_A^-(x*y)\le \max\{v_A^-((x*y)*y),v_A^-(y*y)\}$$

$$=\max\{v_A^-((x*y)*y),v_A^-(0)\}=v_A^-((x*y)*y)\le \max\{v_A^-(a),v_A^-(b)\}$$

(ii) Now let $x,y,z\in X$ be such that $((x*y)*z)*a\le b$. Since A is a positive implicative vague ideal of X, it follows from corollary 4.10

$v_A^+((x*z)*(y*z)\ge v_A^+((x*y)*z)\ge \min\{v_A^+(a),v_A^+(b)\}$ and

$v_A^-((x*z)*(y*z)\le v_A^-((x*y)*z)\le \max\{v_A^-(a),v_A^-(b)\}$. This completes the proof

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