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HERSCOVICI's CONJECTURE ON THE PRODUCTS OF PATHS, COMPLETE GRAPHS

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ABSTRACT

Given a connected graph G, distribute k pebbles on its vertices in some configuration C. Specifically a configuration on a graph G is a function f from V(G) to NU{0} representing an assignment of pebbles on G. We call the total number of pebbles, k, the size of the configuration. A pebbling move is defined as the removal of two pebbles from a vertex and addition of one of those pebbles on an adjacent vertex. The pebbling number of a connected graph G is the smallest number f(G) such that, however f(G) pebbles are distributed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves. The t-pebbling number $f_t(G)$ of a simple connected graph G is the smallest positive integer such that for every distribution of $f_t(G)$ pebbles on the vertices of G, we can move t pebbles to any target vertex by a sequence of pebbling moves. Graham conjectured that For any connected graphs G and H, $f(G\times H) \le f(G)\, f(H)$. Herscovici further conjectured that $f_{st}(G\times H) \le f_s(G)\, f_t(H)$ for any positive integers s and t. In this paper we show that Herscovici's conjecture is true when G is a path, complete graph and H is a graph satisfying the 2t- pebbling property.

Keywords: *Path, complete graph, t-pebbling number, Herscovici's conjecture.*

1.1. INTRODUCTION

The pebbling number is known for many simple graphs including paths, cycle and trees, but is unknown for most graphs and is hard to compute for any given graph that does not fall into one of these classes. Therefore, it is an interesting question if there is an information we can gain about the pebbling number of more complex graphs from the knowledge of the pebbling number of some graphs for which we know. In the first paper on graph pebbling [1] Chung proposed the following conjecture. The conjecture is perhaps the most compelling open question in graph pebbling known as **Grahams conjecture.**

Conjecture 1.1.1 (Graham ([7]): For all graphs G_1 and G_2, we have $f(G_1\times G_2) \le f(G_1)f(G_2)$.

There are a number of results that support Graham's conjecture, the first of which is $f(Q_d)=2^d$. The hypercube is formed by a product of length two paths; $Q_d=Q_{d-1}\times P_2$. And we know that $2^d=f(Q_d)=f(Q_{d-1})f(P_2)=2^{d-1}2$. In addition to this the result has been shown to be true for the product of trees, the product of some specific cycles, and the product of a complete graph and any graph with the two pebbling property. In proving Graham's conjecture on graph pebbling two properties are used in the literature. They are the 2-pebbling property[7] and the odd 2-pebbling property. In [3], Lourdusamy has defined the 2t-pebbling property of a graph.

Definition 1.1.2 ([3]): Given t-pebbling number of G, let p be the number of pebbles of G, let q be the number of vertices with at least one pebble. We say that G satisfies the 2t-pebbling property if it is possible to move 2t pebbles to any specified target vertex of G starting from every configuration in which $p \ge 2f_t(G)-q+1$ or equivalently $p+q > 2f_t(G)$ for all t.

The direct product of two graphs is defined as follows:

Definition 1.1.3 [2]: If $G=(V_G, E_G)$ and $H=(V_H, E_H)$ be two graphs, the direct product of G and H is the graph G×H whose vertex set is the cartesian product $V_{GxH}=V_G\times V_H=\{(x, y);\ x\in V_G,\ y\in V_H)$ and whose edges are given by $E_{GxH}=\{(x, y), (x', y');\ x = x' \text{ and } (y, y') \in E_H \text{ or } (x, x') \in E_G \text{ and } y = y'\}$

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We find the following theorems in [6]

Theorem 1.1.4 [6]: Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t+n-2$.

Theorem 1.1.5 [6]: The t-pebbling number of the path on n vertices is given by $f_t(P_n) = t2^{n-1}$. With regard to the 2t-pebbling property we find the following results in [3], [4], [5], [6] and [7].

Theorem 1.1.6 [7]: All diameter two graphs satisfy the two pebbling property.

Theorem 1.1.7 [3]: All paths satisfy the 2t-pebbling property for all t.

Theorem 1.1.8 [4]: Let K_n be a complete graph on n vertices. Then K_n satisfies the 2t-pebbling property for all t.

Theorem 1.1.9 [5]: The star graph $K_{1,n}$ where $n > 1$ satisfies the 2t-pebbling property.

Theorem 1.1.10 [6]: Fan graphs satisfy the 2t-pebbling property.

With regard to the t-pebbling conjecture on products of graphs we find the following Theorems in [3], and [5].

Theorem 1.1.11 [3] Let P_m be a path on m vertices. If G satisfies the 2t-pebbling property, then $f_t(P_m \times G) \leq 2^{m-1} f_t(G)$ for all t.

Theorem 1.1.12[5]: Let K_n be a complete graph on n vertices where $n \geq 2$ and let G be a graph with the 2t-pebbling property. Then $f_t(K_n \times G) \leq f(K_n) f_t(G)$ for all t.

A.Lourdusamy [3] generalizes Graham's conjecture as follows:

Conjecture 1.1.13 (Lourdusamy [3]): For any connected graphs G and H we have $f_t(G \times H) \leq f(G)\, f_t(H)$ for all t.

This conjecture is called the t –pebbling conjecture and Lourdusamy proved it when G is an even cycle and H satisfies a variation of the two-pebbling property. Herscovici conjectured as

Conjecture1.1.14: For any connected graphs G and H, We have $f_{st}(G \times H) \leq f_s(G)\, f_t(H)$ for all s, t.

In this paper, we prove that Herscovici's conjecture is true when G is a path, complete graph and H is a graph having the 2t pebbling property.

1.2. HERSCOVICI'S CONJECTURE ON PRODUCTS OF PATHS

Theorem 1.2.1: Let P_2 be the path on two vertices x_1 and x_2 and suppose G satisfies the 2t-pebbling property. Then $f_{st}(P_2 \times G) \leq f_s(P_2) f_t(G)$ for all s, t.

Proof: Without loss of generality assume that the target vertex is (x_1, y) for some y. By using induction on s we will prove the Theorem. For s =1 the theorem is true by Theorem 1.1.11. Assume s > 1. There are at least $4f_t(G)$ pebbles on P_2xG. By using at most $2f_t(G)$ pebbles, we can put $f_t(G)$ pebbles on $\{x_1\} \times G$. Hence t pebbles can be moved to (x_1, y). This leaves us with at least $2(s-1)\, f_t(G)$ pebbles which would suffice to put(s-1)t additional pebbles on (x_1, y).

Theorem 1.2.2: Let P_m be a path on m vertices. If G satisfies the 2t-pebbling property then $f_{st}(P_m \times G) \leq s.2^{m-1} f_t(G)$ for all s, t.

Proof: Let P_m: $x_1x_2 \ldots x_m$ be a path on m vertices where $m \geq 2$. The proof is by induction on s. For s = 1, the theorem is true by Theorem 1.1.11. Assume s > 1. There are at least $2^m f_t(G)$ pebbles on $P_m \times G$. Using at most $2.2^{m-1} f_t(G)$ pebbles we can move 2t-pebbles to the target vertex. This leaves us with at least $(s-2)2^{m-1} f_t(G)$ pebbles which would suffice to put (s-2) t additional pebbles on the target.

Corollary 1.2.3: Let P_n be a path on n vertices and $K_{1,m}(m > 1)$ be a star. Then $f_{st}(P_n \times K_{1,m}) \leq f_s(P_n) f_t(K_{1,m})$.

Proof: The corollary follows from Theorem 1.2.2 and Theorem 1.1.9.

Corollary 1.2.4: Let P_n be a path on n vertices and F_m be a fan graph on m vertices. Then $f_{st}(P_n \times F_m) \leq f_s(P_n) f_t(F_m)$ for all s, t

Proof: The corollary follows from Theorem 1.2.2. and Theorem 1.1.10.

Corollary 1.2.5: Let P_n be a path on n vertices and K_m be a complete graph on m vertices Then $f_{st}(P_n \times K_m) \leq f(P_n)f_t(K_m)$ for all s, t.

Proof: The corollary follows from Theorem 1.2.2 and Theorem 1.1.8.

Corollary 1.2.6: Let P_n be a path on n vertices. Then $f_{st}(P_m \times P_n) \leq f_s(P_m)\, f_t(P_n)$ for all s, t.

Proof: The Corollary follows from Theorem 1.2.2 and Theorem 1.1.7.

Thus we have proved that conjecture 1.1.19 is true for all the products of a path by a (i) Path (ii) Fan (iii) Complete graph (iv) Star.

1.3. HERSCOVICI'S CONJECTURE ON PRODUCTS OF COMPLETE GRAPHS

Theorem 1.3.1: Let K_m be a complete graph on m vertices where $m \geq 2$ and let G be a graph with the 2t-pebbling property. Then $f_{st}(Km \times G) \leq f_s(K_m)f_t(G)$ for all s, t.

Proof: We will prove the theorem by induction on s. For s=1 the theorem is true by Theorem 1.1.12. Suppose $(2s+(m-2)\, f_t(G)$ pebbles are distributed on the vertices of $K_m \times G$. Without loss of generality, assume the target is (x_1, y) for some y. Let a_i represent the number of pebbles on $\{x_i\} \times G$. We take s>1. Then there are at least $(m+2)\, f_t(G)$ pebbles on $K_m \times G$. By pigeonhole principle at least one of $\{x_i\} \times G$ $(i \neq 1)$ receives $2f_t(G)$ pebbles. Hence $f_t(G)$ pebbles can be moved to $\{x_i\} \times G$. Hence t pebbles can be moved to our target vertex. This leaves us with at least $(2(s-1) + (m-2))\, f_t(G)$ pebbles which would suffice to put (s-1)t additional pebbles on (x_1, y).

Corollary 1.3.2: Let K_m be complete graph on m vertices and P_n be a path on n vertices. Then $f_{st}(K_m \times P_n) \leq f_s(K_m)\, f_t(P_n)$ for all s, t.

Proof: The corollary follows from Theorem 1.3.1 and Theorem 1.1.7.

Corollary 1.3.3: Let K_m be a complete graph on m vertices and F_n be a fan graph on n vertices. Then
$f_{st}(K_m \times F_n) \leq f_s(K_m)\, f_t(F_n)$ for all s, t.

Proof: The corollary follows from Theorem 1.3.1 and Theorem 1.1.10.

Corollary 1.3.4: Let K_m be a complete graph on m vertices and let $K_{1,n}$ $(n > 1)$ be a star. Then
$f_{st}(K_m \times K_{1,n}) < f_s(K_m)\, f_t(K_{1,n})$ for all s, t.

Proof: The corollary follows from Theorem 1.3.1 and Theorem 1.1.9.

Corollary 1.3.5: Let K_m be a complete graph on m vertices. Then $f_{st}(K_m \times K_n) \leq f_s(K_m)\, f_t(K_n)$ for all s, t.

Proof: The corollary follows from Theorem 1.3.1 and Theorem 1.1.8.

Thus we have proved that conjecture 1.1.14 is true for all the products of a complete graph by a (i) Path (ii) Fan (iii) Complete graph (iv) Star.
Is Herscovici's conjecture is true when G is a Fan graph?

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