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SOME COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACE

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ABSTRACT

The intent of this paper is to initiate the concept of weak-compatibility and semi-compatibility in the context of fuzzy metric spaces. The follow-up investigations by many other mathematicians in due course established a lot of interesting results. Picked up some ideas from these results we established some common fixed point theorem on fuzzy metric space for four mappings which is the generalization of results of Som [2] and Mukherjee [1].

MAIN RESULT

Theorem 1: Let A, B, S and T be self mappings of a complete fuzzy metric space (X, M, *) satisfying

(a) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(b) one of A or S is continuous,
(c) the pair (A, S) is semi-compatible and (B, T) is weak-compatible,
(d) $aM(Ax, By, t) - bM(Sx, Ty, t) \geq \phi\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, By, t), M(Ty, Ax, t), M(Ty, By, t)\}$,

where $\phi : (R^+)^5 \to R^+$ is continuous and strictly increasing in each co-ordinate variable such that for all $x, y \in X$, $a < b + 1$ and for any $\upsilon < 1$, $\phi(\upsilon, \upsilon, a, \upsilon, a_2\upsilon, \upsilon) > \upsilon$, $a_1 + a_2 = 3$. Then A, B, S and T have a unique common fixed point in X.

Proof: Let x_o be any arbitrary point. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ then there exists $x_1, x_2 \in X$ such that

$$Ax_o = Tx_1 = y_1,\ Bx_1 = Sx_2 = y_2.$$

Inductively, construct two sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1},$$
$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2};\quad n = 0, 1, 2, 3, \ldots$$

Let $M_n = M(y_n, y_{n+1}, t)$; $n = 0, 1, 2, 3, \ldots$

We claim that $\{M_n\}$ is a increasing sequence, suppose on the contrary that $M_{2n} > M_{2n+1}$, for some n.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (d), we get

$$aM(Ax_{2n}, Bx_{2n+1}, t) - bM(Sx_{2n}, Tx_{2n+1}, t) \geq \phi\{M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n}, t), M(Tx_{2n+1}, Ax_{2n+1}, t)\}.$$

$$\Rightarrow\ aM(y_{2n+1}, y_{2n+2}, t) - bM(y_{2n}, y_{2n+1}, t) \geq \phi\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, Y_{2n+1}, t), M(y_{2n}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+1}, t), M(y_{2n+1}, Y_{2n+2})\}$$

$$\Rightarrow\ aM_{2n+1} - b\,M_{2n} \geq \phi\{M_{2n}, M_{2n}, M_{2n} + M_{2n+1}, 1, M_{2n+1}\}$$
$$> \phi\{M_{2n+1}, M_{2n+1}, 2M_{2n+1} + M_{2n+1}, M_{2n+1}\}$$
$$> M_{2n+1}$$
$$\Rightarrow\ M_{2n+1} > \frac{b}{a-1} M_{2n}$$
$$\Rightarrow\ M_{2n+1} > M_{2n} \qquad [\because a < b + 1]$$

which is a contradiction.

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Thus $\{M_n\}$ is increasing sequence of positive real number in [0, 1] and therefore $\lim_{n\to\infty} M_n = 1.$

Now, we show that $\{y_n\}$ is a cauchy sequence. Since $\lim_{n\to\infty} M_n = 1,$ it is sufficient to show that $\{y_{2n}\}$ is a cauchy sequence.

Suppose that it is not so, then there is an $\varepsilon > 0$ such that for each integer 2k (k = 0, 1, 2, ...) there exists even integer 2nk and 2mk with 2k < 2nk < 2mk such that

$$M(y_{2nk}, y_{2mk}, t) \le 1 - \varepsilon; \text{ for some } t > 0. \tag{1}$$

Let for each even integer 2k, 2mk be the least positive integer exceeding 2nk satisfying (1), then

$$M(y_{2nk}, y_{2mk-2}, t) > 1 - \varepsilon \text{ and}$$
$$M(y_{2nk}, y_{2mk}, t) \le 1 - \varepsilon. \tag{2}$$

As such, for each even integer 2k, we have

$$1 - \varepsilon > M(y_{2nk}, y_{2mk}, t) \ge M(y_{2nk}, y_{2mk-2}, t/3) * M(y_{2mk-2}, y_{2mk-1}, t/3) * M(y_{2mk-1}, y_{2mk}, t/3).$$

So by (2) and as $k \to \infty$, we get

$$\lim_{n\to\infty} M(y_{2nk}, y_{2mk}, t) = 1 - \varepsilon. \tag{3}$$

Now, using (3) in the triangular inequalities

$$M(y_{2nk}, y_{2mk-1}, t) \ge M(y_{2nk}, y_{2mk}, t/2) * M(y_{2mk}, y_{2mk-1}, t/2)$$

and

$$M(y_{2nk+1}, y_{2mk-1}, t) \ge M(y_{2nk+1}, y_{2nk}, t/3) * M(y_{2nk}, y_{2mk}, t/3) * M(y_{2mk}, y_{2mk-1}, t/3).$$

Taking $k \to \infty$, then

$$M(y_{2nk+1}, y_{2mk-1}, t) \ge 1 - \varepsilon * 1 = 1 - \varepsilon$$

and

$$M(y_{2nk+1}, y_{2mk-1}, t) \ge 1 * 1 - \varepsilon * 1 = 1 - \varepsilon.$$

Then

$$\begin{aligned}
M(y_{2nk}, y_{2mk}) &\ge M(y_{2nk}, y_{2nk+1}, t/2) * M(y_{2nk+1}, y_{2mk}, t/2)\\
&= M(y_{2nk}, y_{2nk+1}, t/2) * M(Bx_{2nk}, Ax_{2mk-1}, t/2)\\
&> M(y_{2nk}, y_{2nk+1}, t/2) * \frac{1}{a}\varphi\frac{\partial^2\Omega}{\partial u^2}\{M(Sx_{2mk-1}, Tx_{2nk}, t/2),\\
&\quad M(Sx_{2mk-1}, Ax_{2mk-1}, t/2), M(Sx_{2mk-1}, Bx_{2nk}, t/2),\\
&\quad M(Tx_{2nk}, Ax_{2mk-1}, t/2), M(Tx_{2nk}, Bx_{2nk}, t/2\} + \frac{b}{a} M(Sx_{2mk-1}, Tx_{2nk}, t/2)\\
&\ge M(y_{2nk}, y_{2nk+1}, t/2) * \frac{1}{a}\varphi\{M(y_{2mk-1}, y_{2nk}, t/2), M(y_{2mk-1}, y_{2mk}, t/2),\\
&\quad M(y_{2mk-1}, y_{2nk+1}, t/2), M(y_{2nk}, y_{2mk}, t/2), \{M(y_{2nk}, y_{2nk+1}\} + \frac{b}{a} M(y_{2mk-1}, y_{2nk}, t/2).
\end{aligned}$$

On taking $k \to \infty$

$$\begin{aligned}
1 - \varepsilon &\ge \frac{1}{a}\varphi\{1-\varepsilon, 0, 1-\varepsilon, 1-\varepsilon, 0\} + \frac{b}{a}(1-\varepsilon)\\
&> \frac{1}{a}(1-\varepsilon) + \frac{b}{a}(1-\varepsilon) = \frac{1+b}{a}(1-\varepsilon)
\end{aligned}$$

$$\Rightarrow 1 - \varepsilon > 1 - \varepsilon$$

which is a contradiction.

Hence $\{y_{2n}\}$ is a cauchy sequence in X. By completeness of X, $\{y_n\}$ converges to $z \in X$. Hence, the subsequences

$$\{Ax_{2n}\} \to z, \{Sx_{2n}\} \to z, \tag{4}$$

$$\{Tx_{2n+1}\} \to z, \{Bx_{2n+1}\} \to z. \tag{5}$$

Since the limit of a sequence in fuzzy metric space is unique we obtain that

$$Az = Sz$$

Step-1: Now, we will prove that Az = z. Suppose on the contrary $Az \neq z$.

By putting $x = z$, $y = x_{2n+1}$ in (d) we have

$$aM(Az, Bx_{2n+1}, t) - bM(Sz, Tx_{2n+1}, t) \geq \phi\{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$$

$$\Rightarrow \quad aM(Az, z, t) - bM(Az, z, t) \geq \phi\{M(Az, z, t), M(Az, Az, t), M(Az, z, t), M(z, Az, t), M(z, z, t)\}$$
$$\geq \phi\{M(Az, z, t), 1, M(Az, z, t), M(Az, z, t), 1\}$$
$$\geq \phi\{M(Az, z, t), M(Az, z, t), 2M(Az,z, t), M(Az, z, t), M(Az, z, t)\}$$

$$\Rightarrow \quad (a - b)\, M(Az, z, t) > M(Az, z, t)$$

which is a contradiction.

Hence $z = Az = Sz$.

Step-2: Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that

$$z = Az = Tu.$$

Now, we have to prove that z = Bu, suppose on the contrary that $z \neq Bu$

Putting $x = x_{2n}$, $y = u$ in (d) we get.

$$aM(Ax_{2n}, Bu, t) - bM(Sx_{2n}, Tu, t) \geq \phi\{M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Bu, t), M(Tu, Ax_{2n}, t), M(Tu, Bu, t)\}.$$

On taking limit as $n \to \infty$ and using (4) we obtain that

$$aM(z, Bu, t) - bM(z, z, t) \geq \phi\{M(z, z, t), M(z, z, t), M(z, Bu, t), M(z, z, t), M(z, Bu, t)\}$$

$$\Rightarrow \quad aM(z, Bu, t) - b \geq \phi\{1, 1, M(z, Bu, t)\ 1, M(z, Bu, t)\}$$

$$aM(z, Bu, t) - bM(z, Bu, t) > \phi\{M(z, Bu, t), M(z, Bu, t), 2M(z, Bu, t), M(z, Bu, t), M(z, Bu, t)\}$$

$$(a - b)\, M(z, Bu, t) > M(z, Bu, t)$$

which is a contradiction.

Hence z = Bu = Tu and the weak compatibility of (B, T) gives

$$TBu = Btu$$

i.e. $Tz = Bz$

Step-3: By putting $x = z$, $y = z$ in (d) and assuming $Az \neq Bz$, we have.

$$aM(Az, Bz, t) - bM(Sz, Tz, t) \geq \phi\{M(Sz, Tz, t), M(Sz, Az, t), M(Sz, Bz, t), M(Tz, Az, t), M(Tz, Bz, t)\}$$

$$\Rightarrow \quad aM(Az, Bz, t) - b\, M(Az, Bz, t) \geq \phi\{M(Az, Bz, t), M(Az, Az, t), M(Az, Bz, t), M(Bz, Az, t), M(Tz, Tz, t)\}$$

$$\Rightarrow \quad (a-b)\, M(Az, Bz, t) \geq \phi\{M(Az, Bz, t), 1,(M(Az, Bz, t), M(Az, Bz, t), 1\}$$
$$> \phi\{M(Az, Bz, t), M(Az, Bz, t), 2M(Az, Bz, t), M(Az, Bz, t), M(Az, Bz, t)\}$$

$$\Rightarrow \quad (a - b)\, M(Az, Bz, t) > M(Az, Bz, t)$$

Which is a contradiction. Hence Az = Bz.

Combining the result from **Steps 1, 2, 3** we obtain that

$$z = Az = Bz = Sz = Tz$$

Therefore z is a common fixed point of A, B, S and T.

Case-2: S is continuous

As S is continuous and (A, S) is semi-compatible, we have.

$$SAx_{2n} \to Sz,\ S^2x_{2n} \to Sz,\ ASx_{2n} \to Sz \qquad (6)$$

Thus, $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n} = Sz$

We prove Sz = z, suppose on the contrary that $Sz \neq z$.

Step-4: Putting $x = Sx_{2n}$, $y = x_{2n+1}$ in (d)

$$aM(ASx_{2n}, Bx_{2n+1}, t) - bM(SSx_{2n}, Tx_{2n+1}, t) \geq \phi\{M(SSx_{2n}, Tx_{2n+1}, t), M(SSx_{2n}, ASx_{2n}, t), M(SSx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, ASx_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$$

$$\Rightarrow \quad aM(Sz, z, t) - bM(Sz, z, t) \geq \phi\{M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, Sz, t), M(z, z, t)\}$$
$$\geq \phi\{M(Sz, z, t), 1, M(Sz, z, t), M(Sz, z, t), 1\}$$
$$\geq \phi\{M(Sz, z, t), M(Sz, z, t), 2M(Sz, z, t), M(Sz, z, t), M(Sz, z, t)\}$$

$$\Rightarrow \quad (a - b)\, M(Sz, z, t) > M(Sz, z, t)$$

which is a contradiction. Hence Sz = z.

Step-5: By putting $x = z$, $y = x_{2n+1}$ in (d)

$$aM(Az, Bx_{2n+1}, t) - bM(Sz, Tx_{2n+1}, t) \geq \phi\{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$$

$$\Rightarrow \quad aM(Az, z, t) - bM(z, z, t) \geq \phi\{M(z, z, t), M(z, Az, t), M(z, z, t), M(z, Az, t), M(z, z, t)\}$$

$$\Rightarrow \quad aM(Az, z, t) - b \geq \phi\{1, M(Az, z, t), 1, M(Az, z, t), 1)$$

$$\Rightarrow \quad aM(Az, z, t) - b\,(Az, z, t) > \phi\{M(Az, z, t), M(Az, z, t), 2M(Az, z, t), M(Az, z, t), M(Az, z, t)\}$$

$$\Rightarrow \quad (a-b)\, M(Az, z, t) > M(Az, z, t)$$

Which is a contradiction.

Hence Az = z = Sz.

Also Bz = Tz = z follows from **step 1, 2** we get that
z = Az = Bz = Sz = Tz.

Hence z is a common fixed point of A, B, S and T.

UNIQUENESS

Let z_1 and z_2 be two common fixed points of the A, B, S and T.

Then $z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1$ and $z_2 = Az_2 = Bz_2 = Sz_2 = Tz_2$.

Suppose $z_1 \neq z_2$. From (d), we have

$$aM(Az_1, Bz_2, t) - bM(Sz_1, Tz_2, t) \geq \phi\{M(Sz_1, Tz_2, t), M(Sz_1, Az_1, t), M(Sz_1, Bz_2, t), M(Tz_2, Az_1, t), M(Tz_2, Bz_2, t)\}$$

$$\Rightarrow \quad aM(z_1, z_2, t) - bM(z_1, z_2, t) \geq \phi\{M(z_1, z_2, t), M(z_1, z_1, t), M(z_1, z_2, t), M(z_2, z_1, t), M(z_2, z_2, t)\}$$
$$\geq \phi\{M(z_1, z_2, t), 1, M(z_1, z_2, t), M(z_2, z_1, t), 1\}$$
$$> \phi\{M(z_1, z_2, t), M(z_1, z_2, t), 2M(z_1, z_2, t), M(z_2, z_2, t), M(z_1, z_2, t)\}$$

$$\Rightarrow \quad (a-b)\, M(z_1, z_2, t) > M(z_1, z_2, t)$$

which is a contradiction. Hence $z_1 = z_2$.

Thus z is a unique common fixed point of A, B, S and T.

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