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STRONGLY MAGIC SQUARES AS RINGS AND FIELDS

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ABSTRACT

A generic definition for the well known class of magic squares-strongly magic square is given and ring structure of strongly magic squares is discussed. In this paper strongly magic squares are proved to have a ring structure and some particular strongly magic squares form commutative ring with unity. The paper also covers field structure of strongly magic squares.

Keywords: *Magic Square, Magic Constant, Strongly Magic Square (SMS), Abelian Groups, Rings, Fields*

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I. INTRODUCTION

A normal magic square is a square array of consecutive numbers from $1 \dots n^2$where the rows and column add up to the same number [1]. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, SMS of order 4 will have a stronger property that the sum of the entries of the 2×2 subsquares taken without any gaps between the rows or columns is also the magic constant [2]. There are many recreational aspects of stronglymagic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possessadvanced mathematical structures.

II. NOTATIONS AND MATHEMATICALPRELIMINARIES

(A) Magic Square: A magic square of order n over a field R is an n[th] order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho for i = 1,2, \dots..n \quad (1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho for i = 1,2, \dots..n \quad (2)$$

$$\sum_{i=1}^{n} a_{ii} = \rho, \quad \sum_{i=1}^{n} a_{i,n-i+1} = \rho \quad (3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [3].

(B) Magic Constant: The constantρin the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

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(C) Strongly magic square (SMS): Generic Definition: A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad for \quad i = 1,2,\ldots..n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad for \quad i = 1,2,\ldots..n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho, \quad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \tag{6}$$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \ for \ i,j = 1,2,\ldots..n^2 \tag{7}$$

$where\ the\ subscripts\ are\ Congruent\ modulo n^2$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The n^{th} order subsquare sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

(D) Group: A group $(G,*)$ is a nonempty set G closed under a binary operation $*$ such that the following axioms are satisfied

(i) $*$ is associative in G. ie, $a * (b * c) = (a * b) * c, \forall\ a, b, c \in G$

(ii) $\exists\ e \in G$, such that $e * a = a * e, \forall\ a \in G$, where e is the identity element.

(iii) Corresponding to each $a \in G, \exists\ b \in G$ such that $a * b = b * a = e$, where b is the inverse of a [4,5]

(E) Abelian Group: A group G is abelian if its binary operation $*$ is commutative. [4]

(F) Rings: A non-empty set R together with two binary operations $+$ and $\cdot$ called addition and multiplication respectively is called a ring denoted as $< R, +, \cdot >$ if the following axioms are satisfied.

i. $< R, + >$ is an abelian group.

ii. Multiplication is associative., i.e., $a.(b.c) = (a.b).c\ \forall\ a, b, c \in R$

iii. Multiplication is distributive with respect to the addition, i.e., $a.(b + c) = a.b + a.c$ (Left distributive law) and $(b + c).a = b.a + c.a$ (Right distributive law) [4]

(G) Commutative Ring: A ring in which the multiplication is commutative is called a commutative ring. A ring with a multiplicative identity element 1 is called a ring with unity. [4]

(H) Field: A ring R with at least two elements is called a field if it

i. is commutative

ii. has unity

iii. is such that each non zero element possesses multiplicative inverse.[4]

(I) Other Notations:

1. SM_s denote the set of all strongly magic squares of order $n^2 \times n^2$
2. $SM_{S(a)}$ denote the set of all strongly magic squares of the form $[a_{ij}]_{n^2 \times n^2}$ such that $a_{ij} = a$ for every $i, j = 1,2, \ldots n^2$. Here A is denoted as $[a]$, i.e. If $A \in SM_{S(a)}$ then $\rho(A) = n^2 a$
3. $SM_{S(0)}$ denote the set of all strongly magic squares of order $n^2 \times n^2$ with magic constant 0, i.e. If $A \in SM_{S(0)}$ then $\rho(A) = 0$

III. PROPOSITIONS AND THEOREMS

Proposition 1: If A and B are two SMS's of order $n^2 \times n^2$ with $\rho(A) = a$ and $\rho(B) = b$, then $C = (\lambda + \mu)(A + B)$ is also a SMS with magic constant $(\lambda + \mu)(\rho(A) + \rho(B))$; for every $\lambda, \mu \in R$

Proof: Let $A = [a_{ij}]_{n^2 \times n^2}$ and $B = [b_{ij}]_{n^2 \times n^2}$

Then$C = (\lambda+\mu)(A+B)$
$= (\lambda+\mu)[a_{ij} + b_{ij}]$
$= [(\lambda+\mu)(a_{ij} + b_{ij})]$

Sum of the i^th row elements of

$$C = \sum_{j=1}^{n^2} c_{ij}$$
$$= \sum_{j=1}^{n^2} [(\lambda+\mu)\,(a_{ij} + b_{ij})]$$
$$= (\lambda+\mu)\left(\sum_{j=1}^{n^2}[a_{ij}] + \sum_{j=1}^{n^2}[b_{ij}]\right)$$
$$= (\lambda+\mu)(a+b)$$
$$= (\lambda+\mu)(\rho(A) + \rho(B))$$

A similar computation holds for column sum

Main diagonal sum

$$\sum_{i=1}^{n^2} c_{ii} = \sum_{i=1}^{n^2}[(\lambda+\mu)(a_{ii} + b_{ii})]$$
$$= (\lambda+\mu)\left(\sum_{i=1}^{n^2}[a_{ii}] + \sum_{i=1}^{n^2}[b_{ii}]\right)$$
$$= (\lambda+\mu)(\mathrm{a}+\mathrm{b})$$
$$= (\lambda+\mu)(\rho(A) + \rho(B))$$

A similar computation holds for co - diagonal sum

The sum of the $n \times n$ sub squares ${M_{kC}}^{(n)}$ is given by

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} c_{i+k,j+l} = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}[(\lambda+\mu)(a_{i+k,j+l} + b_{i+k,j+l})]$$
$$= (\lambda+\mu)\left[\sum_{l=0}^{n-1}\sum_{k=0}^{n-1}[a_{i+k,j+l}] + \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}[b_{i+k,j+l}]\right]$$
$$= (\lambda+\mu)(a+b)$$
$$= (\lambda+\mu)(\rho(A) + \rho(B))$$

From the above propositions the following results can be obtained.

Results: If for every $\lambda, \mu \epsilon R\ and\ A, B \in SM_s$,

1.1) $\lambda\,(A+B) \in SM_s with \rho\,(\lambda\,(A+B)) = \lambda\,(\rho(A) + \rho(B))$

$\boldsymbol{Proof}$: In the above proposition1 put $\mu = 0$

1.2) $(A+B) \in SM_s with \rho\,(\,(A+B)) = \,(\rho(A) + \rho(B))$

$\boldsymbol{Proof}$: By putting $\lambda = 1$ in result 1.1 this can be obtained

1.3) $\lambda A \in SM_s with\ \rho(\lambda A) = \lambda\rho(A)$

$\boldsymbol{Proof}$: It can be easily verified by putting $\mathrm{B} = 0$ in result 1.1

1.4) $-A \in SM_s\ with\ \rho(-A) = -\rho(A)$

$\boldsymbol{Proof}$: $By\ putting\ \lambda = -1$ in result 1.3, it can be obtained

Theorem 2: $< SM_s ,+> for\ msanabelian\ group.$

Proof:

I. **Closure property:** $if A, B \in SM_s, then\ A + B \in SM_s.$ $(from\ above\ result$ 1.2)

II. **Associatively:** $if A, B , C \in SM_s , then A + (B + C) = (A + B) + C \in SM_s$ (Since matrix addition is associative.)

III. **Existence of Identity:** There exists 0 matrix in $SM_s\ so\ that\ A + 0 = 0 + A = A,$ $where\ 0\ acts\ as\ the\ identity\ element.$

IV. Existence of additive inverse: For every $A \in SM_s,\ there\ exists - A \in SM_s\ so\ that\ A + (-A) = 0$ $where\ 0 \in SM_s\ (from result$ 1.4).

V. **Commutatively:** $If A, B \in SM_s, then\ A + B = B + A \in SM_s$ (Since matrix addition is commutative.)

Theorem 3: $< SM_{S(a)}, +> forms\ an\ abelian\ group$

Proof: First we will prove that $SM_{S(a)} is\ a\ subgroup\ of\ the\ abelian\ group\ SM_s.$

It is clear that $SM_{S(a)} \subset SM_s.$

$For A, B \in SM_{S(a)}; A = [a]\ and\ B = [b], \quad then\ clearly\ A - B = [a - b] \in SM_{S(a)}$

Proposition 4: $SM_{S(a)}$ is closed under matrix multiplication.

Proof: Let $AB = D = [d_{ij}] for A = [a]\ and\ B = [b] \in SM_{S(a)}, then\ d_{ij} = \sum_{k=1}^{n^2} a_{ik} . b_{kj} . = n^2 ab.$

Sum of the i th row elements of $D = \sum_{j=1}^{n^2} d_{ij} = \sum_{j=1}^{n^2} n^2 ab = n^4 ab$

Sum of the j th column elements of $D = \sum_{j=1}^{n^2} d_{ij} = \sum_{j=1}^{n^2} n^2 ab = n^4 ab$

Sum of the diagonal elements of $D = \sum_{i=1}^{n^2} d_{ii} = \sum_{i=1}^{n^2} n^2 ab = n^4 ab$

Sum of the co-diagonal elements of $D = \sum_{i=1}^{n^2} d_{i,n^2-i+1} = \sum_{i=1}^{n^2} n^2 ab = n^4 ab$

Sum of the $n \times n$ sub squares of D, $M_{0C}{}^{(n)} or M_{0R}{}^{(n)} is$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} d_{i+k,j+l} = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}[(a_{i+k,j+l} . b_{i+k,j+l})]$$

$$= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1} n^2 ab = n^4 ab.$$

Thus $D = AB \epsilon SM_{S(a)}.$

Theorem 5: $< SM_{S(a)} , +, . > will\ form\ a\ ring$

Proof: Since

a) $SM_{S(a)}$ is an abelian group under matrix addition (By Theorem 3)
b) Matrix multiplication is associative and distributive over addition.
c) $SM_{S(a)}$ is closed under matrix multiplication. (By Proposition 4)

Proposition 6: Let $A = [a],\ B = [b] \in SM_{S(a)},\ then\ A.B = B.A$

Proof: Since $A, B \in SM_{S(a)},\ A.B = [n^2 ab] = [n^2 ba] = B.A$

Theorem 7: $< SM_{S(a)} , +, . > is\ a\ commutative\ ring\ with\ unity\ I_S = \left[\frac{1}{n^2}\right]$

Proof: By Theorem 5 and Proposition 6 $< SM_{S(a)}, +, . > is\ a\ commutative\ ring$

To prove that $I_S = \left[\frac{1}{n^2}\right]$ is the unity it is enough to prove that $A . I_S = A = I_S . A; for\ A \in SM_{S(a)}.$

Clearly $I_S = \left[\frac{1}{n^2}\right] \in SM_{S(a)}$.

Now $A \,.\, I_S = [a].\left[\frac{1}{n^2}\right] = \left[n^2.\frac{a}{n^2}\right] = [a] = I_S.A$

Proposition 8: Let $A, B \in SM_s$, then $A.B$ *does not belongs to* SM_s.

Proof: It can be verified by giving a suitable example.

Let $A = \begin{bmatrix} 16 & 5 & 4 & 9 \\ 2 & 11 & 14 & 7 \\ 13 & 8 & 1 & 12 \\ 3 & 10 & 15 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 13 & 2 & 16 \\ 10 & 8 & 11 & 5 \\ 15 & 1 & 14 & 4 \\ 6 & 12 & 7 & 9 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 212 & 360 & 206 & 378 \\ 368 & 212 & 370 & 206 \\ 206 & 378 & 212 & 360 \\ 370 & 206 & 368 & 212 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 116 & 356 & 450 & 234 \\ 324 & 231 & 315 & 286 \\ 277 & 288 & 332 & 259 \\ 332 & 243 & 303 & 278 \end{bmatrix}$$

which is not a Strongly Magic Square.

This will lead to the following theorem.

Theorem 9: $< SM_S , +, . >$ will not form a ring

Proof: The above proposition 8 shows that matrix multiplication of strongly magic squares is not closed.

Therefore SM_S will not form a ring.

Proposition 10: If $A \in SM_{S(a)}$, then A has a multiplicative inverse in $SM_{S(a)}$. (Here $A \neq 0$)

Proof: Let $A \in SM_{S(a)}$, *then* $A = [a]$. Now we have to find out an element $B \in SM_{S(a)}$ such that $A.B = I_S$, *the identity element of* $SM_{S(a)}$

Take $B = \left[\frac{1}{n^4 a}\right]$ then clearly $B \in SM_{S(a)}$ *and* $A.B = [a].\left[\frac{1}{n^4 a}\right] = \left[\frac{1}{n^4 a}\right].[a] = B.A = \left[\frac{1}{n^2}\right] = I_S$.

Theorem 11: $< SM_{S(a)} , +, . >$will form a field.

Proof: Since $< SM_{S(a)} , +, . >$forms a commutative ring with unity (Theorem 7) and it has amultiplicative inverse (Proposition 10), it will form a field.

IV. CONCLUSION

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different algebra courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described.

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