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ON THE WIENER AND HYPER WIENER INDICES OF CERTAIN DOMINATING GRAPH OPERATIONS OF KRAGUJEVAC TREES

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ABSTRACT

In this paper, we compute the results on certain dominating graph operations (d-transformations) of Kragujevac tree T with respect to distance-based indices, with emphasis on the Wiener index $W(G)$ which is defined as the sum of distances between all pairs of vertices of G and the hyper-Wiener index $WW(G)$ is defined as

$$WW(G)=\frac{1}{2}\sum_{(u,v)\subseteq V(G)}[d_G(u,v)+d_G(u,v)^2].$$

Keywords: *distance, Wiener index, hyper-Wiener index, dominating graph operations.*

Subject Classification: *05C05, 05C07, 05C69.*

1. INTRODUCTION

Throughout this paper, the graphs considered will be assumed to be simple and connected. Let $G=(V,E)$ be such a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)|=n$ and $|E(G)|=m$. By the *open neighborhood* of a vertex v of G we mean the set $N_G(v)=\{u\in V(G):uv\in E(G)\}$. The *d*istance $d_G(u,v)$ between the vertices u and v of the graph G is equal to the length of a shortest path that connects u and v. Other undefined notation and terminology can be found in [8].

A *topological index* of a graph is a single unique number characteristic of the graph and is mathematically invariant under graph automorphism. Usage of topological indices in biology and chemistry began in 1947 when H. Wiener [16, 17] introduced *Wiener index* which is denoted by W and is given by

$$W(G)=\sum_{(u,v)\subseteq V(G)} d_G(u,v)$$

The *hyper-Wiener index* of acyclic graphs was introduced by Milan Randic in 1993. Then Klein *et al.*[13], generalized Randic's definition for all connected graphs, as a generalization of the Wiener index. It is defined as

$$WW(G)=\frac{1}{2}\sum_{(u,v)\subseteq V(G)}[d_G(u,v)+d_G(u,v)^2]$$

Further, information about Wiener index and hyper-Wiener index are given in [4, 5, 6, 11, 12, 18].

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2. KRAGUJEVAC TREES

A connected acyclic graph is called a *tree*. A *rooted tree* is a tree in which one particular vertex is distinguished; this vertex is referred to as the root. The vertex of degree one is a *pendant vertex*. The vertex adjacent to pendant vertex is called *support vertex*. The formal definition of a Kragujevac tree was introduced in [7].

Definition 1 [7]: *Let* P_3 *be the 3-vertex tree,rooted at one of its terminal vertices. For* $k = 2,3,...$, *construct the rooted tree* B_k *by identifying the roots of k copies of* P_3. *The vertex obtained by identifying the roots* P_3*-trees is the root of* B_k
.

Examples illustrating the structure of the rooted tree B_k depicted in Figure 1.

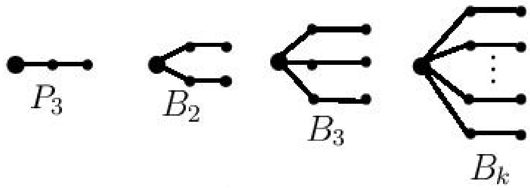


Figure-1.

Definition 2 [7]: *Let* $d \geq 2$ *be an integer. Let* $\beta_1, \beta_2, \beta_3 ... \beta_d$ *be rooted trees specified in Definition 1, i.e.,* $\beta_1, \beta_2, \beta_3 ..., \beta_d \in \{B_2, B_3, ...\}$. *A Kragujevac tree T is a tree possessing a vertex of degree d, adjacent to the roots of* $\beta_1, \beta_2, \beta_3 ..., \beta_d$. *This vertex is said to be the central vertex of T, where d is the degree of T. The subgraphs* $\beta_1, \beta_2, \beta_3 ..., \beta_d$ *are the branches of T. Recall that some (or all) branches of T may be mutually isomorphic.*

A typical Kragujevac tree of degree d=5 is depicted in Figure 2.

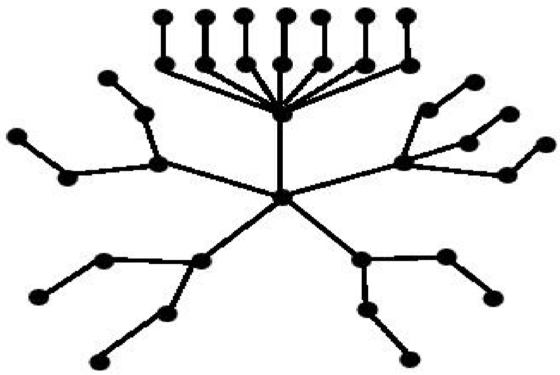


Figure-2

Clearly, the branch B_k has 2k+1 vertices. Therefore, the vertex set and the edge set of the Kragujevac tree T are:

$$V(T) = 1 + \sum_{i=1}^{d}(2k_i + 1) \quad and \quad E(T) = \sum_{i=1}^{d}(2k_i + 1)$$

We denote the vertices of Kragujevac trees as follows:

- pendant vertices by x_i
- support vertices by w_i
- vertices belongs to the set $N(w_i) - \{x_i\}_{i=1}^{k}$ by v_i
- the central vertex by u

Recent information on Kragujevac trees can be found in [3, 7].

3. DOMINATING GRAPH OPERATIONS

The theory of domination has emerged as one of the most studied area in graph theory and its allied branch in mathematics. The wide variety of domination parameters have been defined and studied their applications in various fields [9]. A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The dominating set D is called a *minimal dominating* set if no proper subset of D is a dominating set. A dominating set D is a *total dominating set* if the induced subgraph $< D >$ has no isolated vertices. The *total domination number* $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. For a comprehensive survey of domination in graphs, see [10].

In the mathematical literature several transformation graphs have been considered and constructed. Let $\mathcal{G}$ denote the set of simple undirected graphs. Various important results in graph theory have been obtained by considering some functions $\mathcal{F}: \mathcal{G} \to \mathcal{G}$ or $\mathcal{F}_s: \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \to \mathcal{G}$ called transformations or operations (here each $\mathcal{G}_i = \mathcal{G}$) and by establishing how these operations affect certain properties or parameters of graphs. The complement, the k-th power of graph, line graph and the total graph are well known examples of such transformations or operations. The same concept of transformation has been applied to dominating sets and defined a variety of *dominating transformation(d-transformation)* graphs by Prof. B. Basavanagoud and his students and Prof. V.R.Kulli [1, 2, 14, 15]. In what follows we define the best known representatives of such graphs.

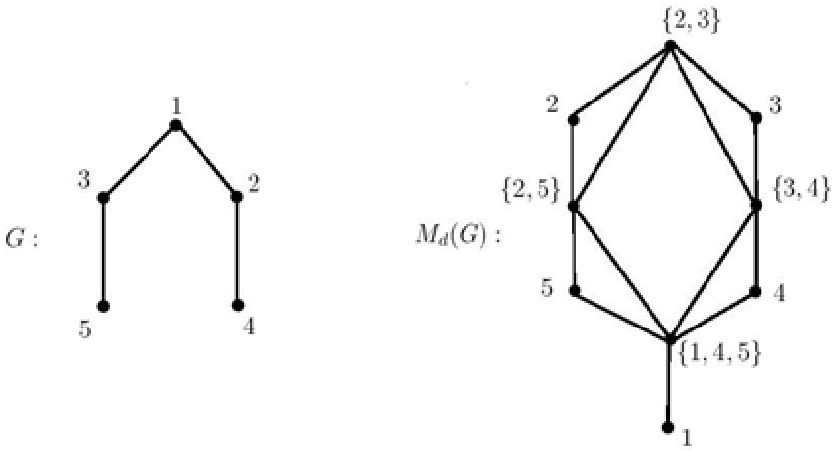
The *middle dominating graph* $M_d(G)$ of a graph G is the graph with vertex set disjoint union of $V \cup A(G)$, where $A(G)$ is the set of all minimal dominating sets of G and (u,v) is an edge if and only if $u \cup v \neq \phi$ whenever $u, v \in A(G) or u \in V, v \in A(G)$ [1].

The *mediate dominating graph* $D_m(G)$ of a graph G is the graph with vertex set $V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V \cup S$ are adjacent if they are not adjacent in G or $u \in V$ and $v = D$ is a minimal dominating set of G containing u [2].

The *total dominating graph* $D_t(G)$ of a graph G is a graph with $V(D_t(G)) = V \cup S$, where S is the set of all minimal total dominating sets of G and with two vertices $u, v \in V(D_t(G))$ adjacent if $u \in V$ and $v = D$ is a minimal total dominating set of G containing u [15].

The *common minimal total dominating graph* $CD_t(G)$ of a graph G is the graph having same vertex set as G with two vertices are adjacent if and only if there exist a minimal total dominating set in G containing them [14].

In Figure 3, a graph G and its graph operations are shown.



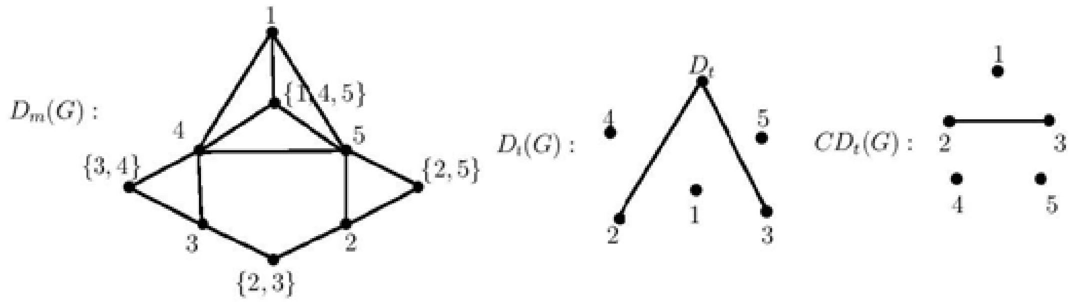
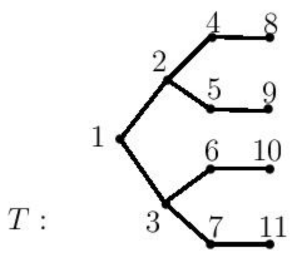


Figure-3: A graph G and its dominating graph operations $M_d(G), D_m(G), D_t(G)$ and $CD_t(G)$

In Figure 4, Kragujevac tree T and its d-transformation graphs $M_d(T)$,$D_m(T)$, $D_t(T)$ and $CD_t(T)$ are given.



$pendant\ vertex\ set\ \{x_i\} = \{8, 9, 10, 11\}$

$support\ vertex\ set\ \{w_i\} = \{4, 5, 6, 7\}$

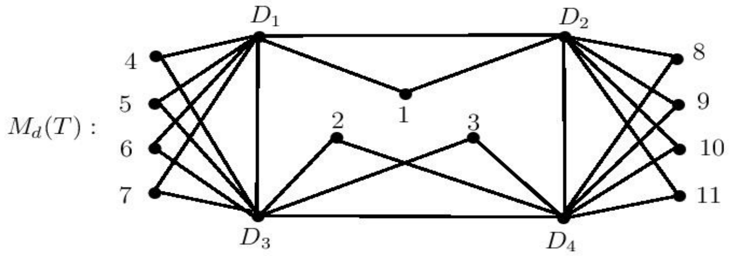
$vertices\ belonging\ to\ set\ \{N(w_i) - \{x_i\}\} = \{v_i\} = \{2, 3\}$

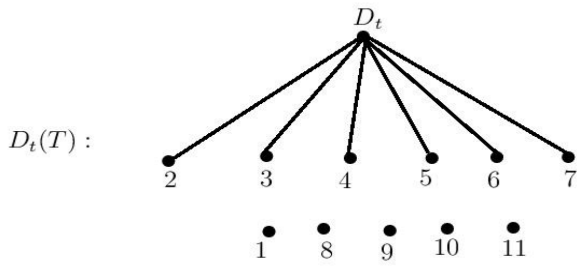
$central\ vertex\ \{u\} = \{1\}$

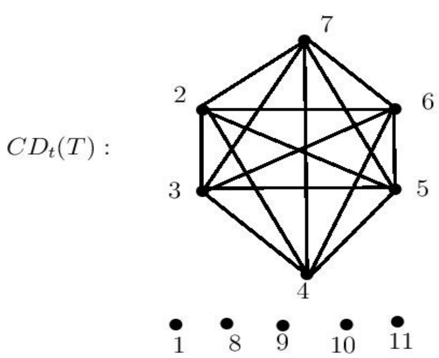
$Dominating\ sets:$

$D_1 = \{1, 4, 5, 6, 7\} \quad D_2 = \{1, 8, 9, 10, 11\}$

$D_3 = \{2, 3, 4, 5, 6, 7\} \quad D_4 = \{2, 3, 8, 9, 10, 11\}$







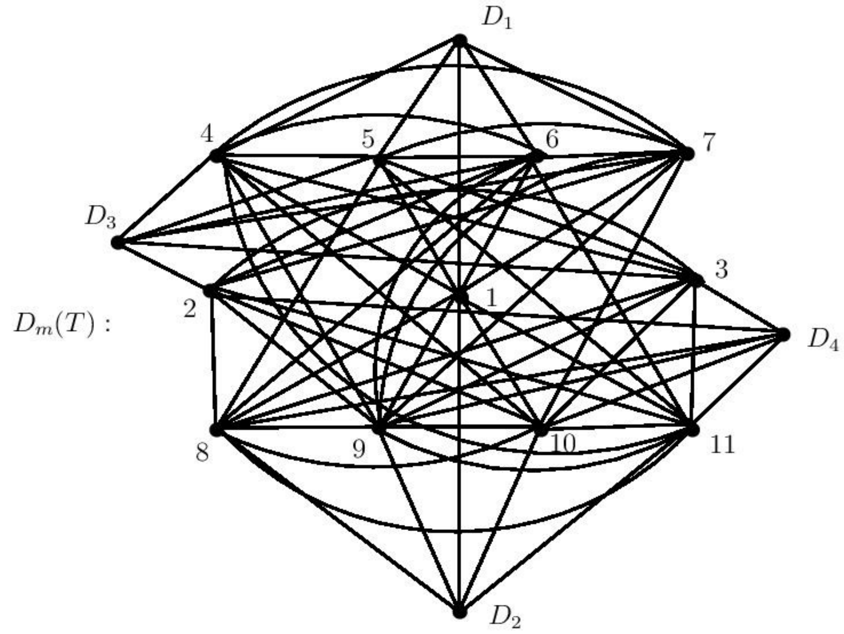


Figure-4: A Kragujevac tree T and its dominating graph operations $M_d(T), D_m(T), D_t(T)$ and $CD_t(T)$

In this paper, expressions for the Wiener and hyper-Wiener indices are obtained for Kragujevac tree T of these graphs.

4. COMPUTING WIENER INDEX FOR M_d(T) ,D_m(T), D_t(T) AND CD_t(T) OF KRAGUJEVAC TREE T.

We begin with the following straightforward, previously known auxiliary result.

Lemma 1: *The dominating sets of Kragujevac tree T are:*

$$D_1 = \{u, \sum_{i=1}^{k} w_i\}$$

$$D_2 = \{u, \sum_{i=1}^{k} x_i\}$$

$$D_3 = \{\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i\}$$

$$D_4 = \{\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} x_i\}$$

Lemma 2: *The total dominating set of the Kragujevac tree T is:*

$$D_t = D_3 = \{\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i\}.$$

Before going to prove our next result we need the following lemma.

Lemma 3: *The distance between each vertex in the middle dominating graph of Kragujevac tree T is:*

1. $d_{M_d(T)}(u, v_i) = 3$
2. $d_{M_d(T)}(u, x_i) = 2$
3. $d_{M_d(T)}(u, w_i) = 2$
4. $d_{M_d(T)}(x_i, v_i) = 2$
5. $d_{M_d(T)}(x_i, w_i) = 3$
6. $d_{M_d(T)}(x_i, x_j) = 2$

7. $d_{M_d(T)}(w_i, w_j) = 2$
8. $d_{M_d(T)}(v_i, v_j) = 2$
9. $d_{M_d(T)}(w_i, v_i) = 2$
10 $d_{M_d(T)}(x_i, \sum_{i=1}^{4} D_i) = 6$
11 $d_{M_d(T)}(u, \sum_{i=1}^{4} D_i) = 6$
12 $d_{M_d(T)}(v_i, \sum_{i=1}^{4} D_i) = 6$
13 $d_{M_d(T)}(w_i, \sum_{i=1}^{4} D_i) = 6$
14 $d_{M_d(T)}(D_i, D_j) = 8.$

Theorem 4: *The Wiener index of the middle dominating graph of Kragujevac tree T is*

$$W[M_d(T)] = k[3k + 4d + 16] + 2[2\ {}^kC_2 + {}^d C_2 + 7] + 9d.$$

Proof: By the definition of $M_d(G)$ it is clear that $V(M_d(T)) = V \cup S$, where $S = \{D_1, D_2, D_3, D_4\}$. By using the definition of Wiener index and Lemma 3, we have

$$W(M_d(T)) = \sum_{(u,v)\in V} d_{M_d(T)}(u,v)$$

$$= \sum_{i=1}^{d} d_{M_d(T)}(u, v_i) + \sum_{i=1}^{k} d_{M_d(T)}(u, x_i) + \sum_{i=1}^{k} d_{M_d(T)}(u, w_i) + \sum_{i=1}^{k} d_{M_d(T)}(x_i, w_i)$$

$$+ \sum_{i=1}^{k}\sum_{i=1}^{d} d_{M_d(T)}(x_i, v_i) + \sum_{i=1}^{k}\sum_{i=1}^{d} d_{M_d(T)}(w_i, v_i) + \sum_{i<j} d_{M_d(T)}(x_i, x_j)$$

$$+ \sum_{i<j} d_{M_d(T)}(w_i, w_j) + \sum_{i<j} d_{M_d(T)}(v_i, v_j) + d_{M_d(T)}(u, \sum_{i=1}^{4} D_i) + \sum_{i=1}^{k} d_{M_d(T)}(x_i, \sum_{i=1}^{4} D_i)$$

$$+ \sum_{i=1}^{k} d_{M_d(T)}(w_i, \sum_{i=1}^{4} D_i) + \sum_{i=1}^{d} d_{M_d(T)}(v_i, \sum_{i=1}^{4} D_i) + \sum_{i<j} d_{M_d(T)}(D_i, D_j)$$

$$= 3d + 2k + 2k + 3k^2 + 2dk + 2dk + 2\ {}^kC_2 + 2\ {}^kC_2 + 2\ {}^dC_2 + 6 + 6k + 6k + 6d + 8$$

$$W[M_d(T)] = k[3k + 4d + 16] + 2[2\ {}^kC_2 + {}^d C_2 + 7] + 9d.$$

Thus, the result follows.

Lemma 5: *The distance between each vertex in the mediate dominating graph of Kragujevac tree* T *is:*

1. $d_{D_m(T)}(u, v_i) = 2$
2. $d_{D_m(T)}(u, x_i) = 1$
3. $d_{D_m(T)}(u, w_i) = 1$
4. $d_{D_m(T)}(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} w_i) = k(k+1)$
5. $d_{D_m(T)}(\sum_{i=1}^{k} x_i, \sum_{i=1}^{d} v_i) = kd$
6. $d_{D_m(T)}(\sum_{i=1}^{k} w_i, \sum_{i=1}^{k} v_i) = k(d+1)$

7. $d_{D_m(T)}(x_i, x_j) = 2\ {}^kC_2$
8. $d_{D_m(T)}(w_i, w_j) = 2\ {}^kC_2$
9. $d_{D_m(T)}(v_i, v_j) = 2\ {}^dC_2$
10. $d_{D_m(T)}(u, \sum_{i=1}^{4} D_i) = 6$
11. $d_{D_m(T)}(x_i, \sum_{i=1}^{4} D_i) = 6$
12. $d_{D_m(T)}(w_i, \sum_{i=1}^{4} D_i) = 6$
13. $d_{D_m(T)}(v_i, \sum_{i=1}^{4} D_i) = 6$
14. $\sum_{i<j} d_{D_m(T)}(D_i, D_j) = 14.$

Theorem 6: *The Wiener index of the mediate dominating graph of Kragujevac tree* T *is*

$$W[D_m(T)] = k[k + 2d + 16] + 4\ {}^kC_2 + 2\ {}^dC_2 + 8d + 20.$$

Proof: By the definition of $D_m(G)$, $V(D_m(T)) = V \cup S$, where $S = \{D_1, D_2, D_3, D_4\}$. By using the definition of Wiener index and Lemma 5, we have,

$$\begin{aligned}
W(D_m(T)) = &\sum_{(u,v)\in V} d_{D_m(T)}(u,v) \\
&= \sum_{i=1}^{d} d_{D_m(T)}(u, v_i) + \sum_{i=1}^{k} d_{D_m(T)}(u, x_i) + \sum_{i=1}^{k} d_{D_m(T)}(u, w_i) \\
&+ \sum_{i=1}^{k} d_{D_m(T)}(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} w_i) + \sum_{i=1}^{k} d_{D_m(T)}(\sum_{i=1}^{k} x_i, \sum_{i=1}^{d} v_i) \\
&+ \sum_{i=1}^{d} d_{D_m(T)}(\sum_{i=1}^{k} w_i, \sum_{i=1}^{k} v_i) + \sum_{i<j} d_{D_m(T)}(x_i, x_j) + \sum_{i<j} d_{D_m(T)}(w_i, w_j) \\
&+ \sum_{i<j} d_{D_m(T)}(v_i, v_j) + d_{D_m(T)}(u, \sum_{i=1}^{4} D_i) + \sum_{i=1}^{k} d_{D_m(T)}(x_i, \sum_{i=1}^{4} D_i) \\
&+ \sum_{i=1}^{k} d_{D_m(T)}(w_i, \sum_{i=1}^{4} D_i) + \sum_{i=1}^{d} d_{D_m(T)}(v_i, \sum_{i=1}^{4} D_i) + \sum_{i<j} d_{D_m(T)}(D_i, D_j) \\
&= d(2) + k(1) + k(1) + (k+1)k + d(k) + k(d+1) + 2\ {}^kC_2 + 2\ {}^kC_2 + 2\ {}^dC_2 + 6 + 6k + 6k \\
&+ 6d + 14
\end{aligned}$$

$$W[D_m(T)] = k[k + 2d + 16] + 4\ {}^kC_2 + 2\ {}^dC_2 + 8d + 20.$$

By the definition of total dominating graph $D_t(T)$, $V(D_t(T)) = V \cup S$, where $S = \{D_3\}$ is a minimal total dominating set of T. Hence the following lemma gives the information about the distance between each vertex of $D_t(T)$ of Kragujevac tree T.

Lemma 7: *The distance between each vertex in the total dominating graph of Kragujevac tree* T *is*

1. $d_{D_t(T)}(u, v_i) = 0$
2. $d_{D_t(T)}(u, x_i) = 0$
3. $d_{D_t(T)}(u, w_i) = 0$

4. $d_{D_t(T)}(x_i, \sum_{i=1}^{d} v_i) = 0$
5. $d_{D_t(T)}(x_i, w_i) = 0$
6. $d_{D_t(T)}(x_i, x_j) = 0$
7. $d_{D_t(T)}(w_i, w_j) = 2\ {}^kC_2$
8. $d_{D_t(T)}(v_i, v_j) = 2\ {}^dC_2$
9. $d_{D_t(T)}(\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i) = 2kd$
10. $d_{D_t(T)}(u, D_3) = 0$
11. $d_{D_t(T)}(w_i, D_3) = k$
12. $d_{D_t(T)}(v_i, D_3) = d$
13. $d_{D_t(T)}(x_i, D_3) = 0.$

Theorem 8: *The Wiener index of the total dominating graph of Kragujevac tree* T *is*

$$W[D_t(T)] = 2[kd + {}^k C_2 + {}^d C_2] + k + d.$$

Proof: By the definition of $D_t(T)$, $V(D_t(T)) = V \cup S$,where $S = \{D_3\}$. By using the definition of Wiener index and Lemma 7, we have,

$$\begin{aligned}
W(D_t(T)) &= \sum_{(u,v)\in V} d_{D_t(T)}(u,v) \\
&= \sum_{i=1}^{d} d_{D_t(T)}(u,v_i) + \sum_{i=1}^{k} d_{D_t(T)}(u,x_i) + \sum_{i=1}^{k} d_{D_t(T)}(u,w_i) \\
&+ \sum_{i=1}^{k} d_{D_t(T)}(x_i, \sum_{i=1}^{d} v_i) + \sum_{i=1}^{k} d_{D_t(T)}(x_i, w_i) \\
&+ \sum_{i=1}^{k} d_{D_t(T)}(x_i, x_j) + \sum_{i<j} d_{D_t(T)}(w_i, w_j) + \sum_{i<j} d_{D_t(T)}(v_i, v_j) \\
&+ d_{D_t(T)}(\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i) + d_{D_t(T)}(u, D_3) + \sum_{i=1}^{k} d_{D_t(T)}(x_i, D_3) \\
&+ \sum_{i=1}^{k} d_{D_t(T)}(w_i, D_3) + \sum_{i=1}^{d} d_{D_t(T)}(v_i, D_3) \\
&= 0+0+0+0+0+0+2\ {}^kC_2 + 2\ {}^dC_2 + 2kd + 0 + k + d + 0
\end{aligned}$$

$$W[D_t(T)] = 2[kd + {}^k C_2 + {}^d C_2] + k + d.$$

By the definition of common minimal total dominating graph $CD_t(T)$, $V(CD_t(T)) = V$. Hence the following lemma gives the information about the distance between each vertex of $CD_t(T)$ of Kragujevac tree T.

Lemma 9: *The distance between each vertex in the common minimal total dominating graph of Kragujevac tree* T *is*

1. $d_{CD_t(T)}(u, v_i) = 0$
2. $d_{CD_t(T)}(u, x_i) = 0$
3. $d_{CD_t(T)}(u, w_i) = 0$
4. $d_{CD_t(T)}(x_i, \sum_{i=1}^{d} v_i) = 0$

5. $d_{CD_t(T)}(x_i, w_i) = 0$
6. $d_{CD_t(T)}(x_i, x_j) = 0$
7. $d_{CD_t(T)}(w_i, w_j) = {}^k C_2$
8. $d_{CD_t(T)}(v_i, v_j) = 1$
9. $d_{CD_t(T)}(\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i) = kd.$

Theorem 10: *The Wiener index of the common minimal total dominating graph of Kragujevac tree* T *is*

$$W[CD_t(T)] = kd + {}^k C_2 + 1.$$

Proof: By the definition of *CD_t(T), V(CD_t(T))=V.* By using the definition of Wiener index and Lemma 8, we have,

$$\begin{aligned}
W(CD_t(T)) &= \sum_{(u,v)\in V} d_{CD_t(T)}(u,v) \\
&= \sum_{i=1}^{d} d_{CD_t(T)}(u, v_i) + \sum_{i=1}^{k} d_{CD_t(T)}(u, x_i) + \sum_{i=1}^{k} d_{CD_t(T)}(u, w_i) \\
&+ \sum_{i=1}^{k} d_{CD_t(T)}(x_i, \sum_{i=1}^{d} v_i) + \sum_{i=1}^{k} d_{CD_t(T)}(x_i, w_i) \\
&+ \sum_{i=1}^{k} d_{CD_t(T)}(x_i, x_j) + \sum_{i<j} d_{CD_t(T)}(w_i, w_j) + \sum_{i<j} d_{CD_t(T)}(v_i, v_j) \\
&+ d_{CD_t(T)}(\sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i) \\
&= 0+0+0+0+0+0+{}^k C_2 + 1 + kd
\end{aligned}$$

$$W[CD_t(T)] = kd + {}^k C_2 + 1.$$

5.COMPUTING HYPER-WIENER INDEX FOR M_d(T), D_m(T), D_t(T) AND CD_t(T) OF KRAGUJEVAC TREE T.

Using the definition of hyper-Wiener index and Lemma 3,5,7 and 9 we can deduce the expression for the hyper-Wiener index of $M_d(T)$, $D_m(T)$, $D_t(T)$ and $CD_t(T)$ of Kragujevac tree T.

By the definition of middle dominating graph $M_d(G)$ and Lemma 3, we can deduce hyper-Wiener index for $M_d(T)$ for Kragujevac tree in the following theorem.

Theorem 11: *The hyper-Wiener index of the middle dominating graph of Kragujevac tree* T *is*

$$\begin{aligned}
WW(M_d(T)) = \frac{1}{2}[&k(9k^3 + 83k + 8d^2k + 4d + 16) + 9d(5d+1) + 4\ {}^kC_2(2\ {}^kC_2 + 1) + 2\ {}^dC_2(2\ {}^dC_2 + 1) \\
&+ 114].
\end{aligned}$$

Proof: By the definition of $M_d(G)$ it is clear that $V(M_d(T)) = V \cup S,$ where $S = \{D_1, D_2, D_3, D_4\}$. By using the definition of hyper-Wiener index and Lemma 3, we have

$$\begin{aligned}
WW(M_d(T)) &= \frac{1}{2}\sum_{(u,v)\in V(G)} [d_{M_d(T)}(u,v) + d_{M_d(T)}(u,v)^2] \\
&= \frac{1}{2}[[\sum_{i=1}^{d} d_{M_d(T)}(u, v_i) + \sum_{i=1}^{d} d_{M_d(T)}(u, v_i)^2] \\
&+ [\sum_{i=1}^{k} d_{M_d(T)}(u, x_i) + \sum_{i=1}^{k} d_{M_d(T)}(u, x_i)^2] + [\sum_{i=1}^{k} d_{M_d(T)}(u, w_i)
\end{aligned}$$

$$+\sum_{i=1}^{k}d_{M_d(T)}(u,w_i)^2]+[\sum_{i=1}^{k}d_{M_d(T)}(x_i,w_i)+\sum_{i=1}^{k}d_{M_d(T)}(x_i,w_i)^2]$$
$$+[\sum_{i=1}^{k}\sum_{i=1}^{d}d_{M_d(T)}(x_i,v_i)+\sum_{i=1}^{k}\sum_{i=1}^{d}d_{M_d(T)}(x_i,v_i)^2]$$
$$+[\sum_{i=1}^{k}\sum_{i=1}^{d}d_{M_d(T)}(w_i,v_i)+\sum_{i=1}^{k}\sum_{i=1}^{d}d_{M_d(T)}(w_i,v_i)^2]$$
$$+[\sum_{i<j}d_{M_d(T)}(x_i,x_j)+\sum_{i<j}d_{M_d(T)}(x_i,x_j)^2]+[\sum_{i<j}d_{M_d(T)}(w_i,w_j)$$
$$+\sum_{i<j}d_{M_d(T)}(w_i,w_j)^2]+[\sum_{i<j}d_{M_d(T)}(v_i,v_j)+\sum_{i<j}d_{M_d(T)}(v_i,v_j)^2]$$
$$+[d_{M_d(T)}(u,\sum_{i=1}^{4}D_i)+d_{M_d(T)}(u,\sum_{i=1}^{4}D_i)^2]+[\sum_{i=1}^{k}d_{M_d(T)}(x_i,\sum_{i=1}^{4}D_i)$$
$$+\sum_{i=1}^{k}d_{M_d(T)}(x_i,\sum_{i=1}^{4}D_i)^2]+[\sum_{i=1}^{k}d_{M_d(T)}(w_i,\sum_{i=1}^{4}D_i)$$
$$+\sum_{i=1}^{k}d_{M_d(T)}(w_i,\sum_{i=1}^{4}D_i)^2]+[\sum_{i=1}^{d}d_{M_d(T)}(v_i,\sum_{i=1}^{4}D_i)$$
$$+\sum_{i=1}^{d}d_{M_d(T)}(v_i,\sum_{i=1}^{4}D_i)^2]+[\sum_{i<j}d_{M_d(T)}(D_i,D_j)+\sum_{i<j}d_{M_d(T)}(D_i,D_j)^2]]$$
$$=\frac{1}{2}[[3d+9d^2]+[2k+4k^2]+[2k+4k^2]+[3k^2+9k^4]+[2dk+4d^2k^2]+[2dk+4d^2k^2]$$
$$+[2\ {}^kC_2+4\,({}^kC_2)^2]+[2\ {}^kC_2+4\,({}^kC_2)^2]+[2\ {}^dC_2+4\,({}^dC_2)^2]+[6+36]+[6k+36k^2]$$
$$+[6k+36k^2]+[6d+36d^2]+[8+64]]$$

$$WW[M_d(T)]=\frac{1}{2}[k(9k^3+83k+8d^2k+4d+16)+9d(5d+1)+4\ {}^kC_2(2\ {}^kC_2+1)+2\ {}^dC_2(2\ {}^dC_2+1)$$
$$+114].$$

Thus, the result follows.

Theorem 12: *The hyper-Wiener index of the mediate dominating graph of Kragujevac tree* T *is*
[[[

$$WW(D_m(T))=\frac{1}{2}[k(k^3+2k^2+77k+2kd^2+2kd+2d+16)+8d(5d+1)+4\ {}^kC_2(2\ {}^kC_2)+1)$$
$$+{}^d\ C_2({}^dC_2+1)+252].$$

Proof: By the definition of $D_m(G), V(D_m(T))=V\cup S$, where $S=\{D_1,D_2,D_3,D_4\}$. By the definition of hyper-Wiener index and Lemma 5, we have

$$WW(D_m(T))=\frac{1}{2}\sum_{(u,v)\in V(G)}[d_{D_m(T)}(u,v)+d_{D_m(T)}(u,v)^2]$$
$$=\frac{1}{2}[[\sum_{i=1}^{d}d_{D_m(T)}(u,v_i)+\sum_{i=1}^{d}d_{D_m(T)}(u,v_i)^2]+[\sum_{i=1}^{k}d_{D_m(T)}(u,x_i)$$
$$+\sum_{i=1}^{k}d_{D_m(T)}(u,x_i)^2]+[\sum_{i=1}^{k}d_{D_m(T)}(u,w_i)+\sum_{i=1}^{k}d_{D_m(T)}(u,w_i)^2]$$
$$+[\sum_{i=1}^{k}d_{D_m(T)}(\sum_{i=1}^{k}x_i,\sum_{i=1}^{k}w_i)+\sum_{i=1}^{k}d_{D_m(T)}(\sum_{i=1}^{k}x_i,\sum_{i=1}^{k}w_i)^2]$$

$$
\begin{aligned}
&+[\sum_{i=1}^{k}d_{D_m(T)}(\sum_{i=1}^{k}x_i,\sum_{i=1}^{d}v_i)+\sum_{i=1}^{k}d_{D_m(T)}(\sum_{i=1}^{k}x_i,\sum_{i=1}^{d}v_i)^2]+[\sum_{i=1}^{d}d_{D_m(T)}(\sum_{i=1}^{k}w_i,\sum_{i=1}^{k}v_i)\\
&+\sum_{i=1}^{d}d_{D_m(T)}(\sum_{i=1}^{k}w_i,\sum_{i=1}^{k}v_i)^2]+[\sum_{i<j}d_{D_m(T)}(x_i,x_j)+\sum_{i<j}d_{D_m(T)}(x_i,x_j)^2]\\
&+[\sum_{i<j}d_{D_m(T)}(w_i,w_j)+\sum_{i<j}d_{D_m(T)}(w_i,w_j)^2]+[\sum_{i<j}d_{D_m(T)}(v_i,v_j)+\sum_{i<j}d_{D_m(T)}(v_i,v_j)^2]\\
&+[d_{D_m(T)}(u,\sum_{i=1}^{4}D_i)+d_{D_m(T)}(u,\sum_{i=1}^{4}D_i)^2]+[\sum_{i=1}^{k}d_{D_m(T)}(x_i,\sum_{i=1}^{4}D_i)\\
&+\sum_{i=1}^{k}d_{D_m(T)}(x_i,\sum_{i=1}^{4}D_i)^2]+[\sum_{i=1}^{k}d_{D_m(T)}(w_i,\sum_{i=1}^{4}D_i)+\sum_{i=1}^{k}d_{D_m(T)}(w_i,\sum_{i=1}^{4}D_i)^2]\\
&+[\sum_{i=1}^{d}d_{D_m(T)}(v_i,\sum_{i=1}^{4}D_i)+\sum_{i=1}^{d}d_{D_m(T)}(v_i,\sum_{i=1}^{4}D_i)^2]+[\sum_{i<j}d_{D_m(T)}(D_i,D_j)\\
&+\sum_{i<j}d_{D_m(T)}(D_i,D_j)^2]\\
&=\frac{1}{2}[[2d+4d^2]+[k+k^2]+[k+k^2]+[(k+1)k+k^2(k+1)^2]+[dk+k^2d^2]+[k(d+1)\\
&+k^2(d+1)^2]+[2\ {}^kC_2+4\,({}^kC_2)^2]+[2\ {}^kC_2+4\,({}^kC_2)^2]+[2\ {}^dC_2+4\,({}^dC_2)^2]\\
&+[6+36]+[6k+36k^2]+[6k+36k^2]+[6d+36d^2]+[14+196]
\end{aligned}
$$

$$
\begin{aligned}
WW(D_m(T))=\frac{1}{2}[&k(k^3+2k^2+77k+2kd^2+2kd+2d+16)+8d(5d+1)+4\ {}^kC_2(2\ {}^kC_2)+1)\\
&+{}^d\,C_2({}^dC_2+1)+252].
\end{aligned}
$$

Thus, the result follows.

Theorem 13: *The hyper-Wiener index of the total dominating graph of Kragujevac tree* T *is*

$$WW(D_t(T))=\frac{1}{2}[2kd[1+2kd]+d[d+1]+k[k+1]+2\ {}^kC_2[1+2\ {}^kC_2]+2\ {}^dC_2[1+2\ {}^dC_2]].$$

Proof: By the definition of hyper-Wiener index and Lemma 7, we have

$$
\begin{aligned}
WW(D_t(T))&=\frac{1}{2}\sum_{(u,v)\in V}[d_{D_t(T)}(u,v)+d_{D_t(T)}(u,v)^2]\\
&=\frac{1}{2}[[\sum_{i=1}^{d}d_{D_t(T)}(u,v_i)+\sum_{i=1}^{d}d_{D_t(T)}(u,v_i)^2]+[\sum_{i=1}^{k}d_{D_t(T)}(u,x_i)\\
&+\sum_{i=1}^{k}d_{D_t(T)}(u,x_i)^2]+[\sum_{i=1}^{k}d_{D_t(T)}(u,w_i)+\sum_{i=1}^{k}d_{D_t(T)}(u,w_i)^2]\\
&+[\sum_{i=1}^{k}d_{D_t(T)}(x_i,\sum_{i=1}^{d}v_i)+\sum_{i=1}^{k}d_{D_t(T)}(x_i,\sum_{i=1}^{d}v_i)^2]\\
&+[\sum_{i=1}^{k}d_{D_t(T)}(x_i,w_i)+\sum_{i=1}^{k}d_{D_t(T)}(x_i,w_i)^2]+[\sum_{i=1}^{k}d_{D_t(T)}(x_i,x_j)\\
&+\sum_{i=1}^{k}d_{D_t(T)}(x_i,x_j)^2]+[\sum_{i<j}d_{D_t(T)}(w_i,w_j)+\sum_{i<j}d_{D_t(T)}(w_i,w_j)^2]\\
&+[\sum_{i<j}d_{D_t(T)}(v_i,v_j)+\sum_{i<j}d_{D_t(T)}(v_i,v_j)^2]+[\sum_{i=1}^{k}d_{D_t(T)}(w_i,D_3)+\sum_{i=1}^{k}d_{D_t(T)}(w_i,D_3)^2]\\
&+[d_{D_t(T)}(\sum_{i=1}^{d}v_i,\sum_{i=1}^{k}w_i)+d_{D_t(T)}(\sum_{i=1}^{d}v_i,\sum_{i=1}^{k}w_i)^2]
\end{aligned}
$$

$$+[d_{D_t(T)}(u,D_3)+d_{D_t(T)}(u,D_3)^2]+[\sum_{i=1}^{k}d_{D_t(T)}(x_i,D_3)+\sum_{i=1}^{k}d_{D_t(T)}(x_i,D_3)^2]$$
$$+[\sum_{i=1}^{d}d_{D_t(T)}(v_i,D_3)+\sum_{i=1}^{d}d_{D_t(T)}(v_i,D_3)^2]$$
$$=0+0+0+0+0+0+[2\ {}^kC_2+4\,({}^kC_2)^2]+[2\ {}^dC_2+4\,({}^kC_2)^2]+[2kd+4k^2d^2]+0$$
$$+[k+k^2]+[d+d^2]+0$$

$$WW(D_{t]}(T))=\frac{1}{2}[2kd[1+2kd]+d[d+1]+k[k+1]+2\ {}^kC_2[1+2\ {}^kC_2]+2\ {}^dC_2[1+2\ {}^dC_2]].$$

Hence the proof.

Theorem 14: *The hyper-Wiener index of the common minimal total dominating graph of Kragujevac tree* T *is*

$$WW(CD_t(T))=\frac{1}{2}[kd[1+kd]+{}^kC_2[1+{}^kC_2]+2].$$

Proof: By the definition of hyper-Wiener index and Lemma 9, we have

$$WW(CD_t(T))=\frac{1}{2}\sum_{(u,v)\in V}[d_{CD_t(T)}(u,v)+d_{CD_t(T)}(u,v)^2]$$
$$=\frac{1}{2}[[\sum_{i=1}^{d}d_{CD_t(T)}(u,v_i)+\sum_{i=1}^{d}d_{CD_t(T)}(u,v_i)^2]+[\sum_{i=1}^{k}d_{CD_t(T)}(u,x_i)$$
$$+\sum_{i=1}^{k}d_{CD_t(T)}(u,x_i)^2]+[\sum_{i=1}^{k}d_{CD_t(T)}(u,w_i)+\sum_{i=1}^{k}d_{CD_t(T)}(u,w_i)^2]$$
$$+[\sum_{i=1}^{k}d_{CD_t(T)}(x_i,\sum_{i=1}^{d}v_i)+\sum_{i=1}^{k}d_{CD_t(T)}(x_i,\sum_{i=1}^{d}v_i)^2]$$
$$+[\sum_{i=1}^{k}d_{CD_t(T)}(x_i,w_i)+\sum_{i=1}^{k}d_{CD_t(T)}(x_i,w_i)^2]+[\sum_{i=1}^{k}d_{CD_t(T)}(x_i,x_j)+\sum_{i=1}^{k}d_{CD_t(T)}(x_i,x_j)^2]$$
$$+[\sum_{i<j}d_{CD_t(T)}(w_i,w_j)+\sum_{i<j}d_{CD_t(T)}(w_i,w_j)^2]+[\sum_{i<j}d_{CD_t(T)}(v_i,v_j)+\sum_{i<j}d_{CD_t(T)}(v_i,v_j)^2]$$
$$+[d_{CD_t(T)}(\sum_{i=1}^{d}v_i,\sum_{i=1}^{k}w_i)+d_{CD_t(T)}(\sum_{i=1}^{d}v_i,\sum_{i=1}^{k}w_i)^2]]$$
$$=\frac{1}{2}[0+0+0+0+0+0+{}^kC_2+({}^kC_2)^2+1+1+kd+(kd)^2]$$

$$WW(CD_{t]}(T))=\frac{1}{2}[kd[1+kd]+{}^kC_2[1+{}^kC_2]+2].$$

Hence the proof.

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