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ORTHOGONALITY OF DERIVATIONS AND BIDERIVATIONS IN SEMIPRIME ACCESSIBLE RINGS

P. S. VIJAYA LAKSHMI*[1], K. SUVARNA[2]

[1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

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ABSTRACT

This paper gives the notion of orthogonality between the derivation and biderivation of a semiprime nonassociative accessible ring. We prove that if R is a 2-divisible semiprime accessible ring, B is a biderivation and D is a derivation of R, then B and D are orthogonal if and only if any one of the following equivalent conditions holds for every $x, y \in R$:

(i) $B(x, y)D(z) + D(x)B(z, y) = 0$
(ii) $D(x)B(x, y) = 0$ or $D(x)B(y, x) = 0$
(iii) $DB = 0$
(iv) DB is a biderivation

Keywords: *Derivation, Biderivation, Orthogonality, Semiprime, flexible, accessible rings.*

1. INTRODUCTION

Bresar and Vukman [1], introduced the notion of orthogonality for a pair d and g of derivations on semiprime associative rings and they have proved several necessary and sufficient conditions for d and g to be orthogonal. Daif. *et al.* [2], studied the orthogonality between the derivation and biderivation of a ring and also in terms of a nonzero ideal of 2- divisible semiprime associative rings.

In this paper, we give four conditions equivalent to the notion of orthogonality between the derivation and biderivation of semiprime nonassociative rings, namely, accessible rings. We prove that if R is a 2- divisible semiprime accessible ring, if f and g are derivations of R such that $(f(x)r)\, g(y) = 0 = g(y)\, (rf(x))$, then $f(x)\, (rg(y) = 0 = (g(y)r)\, f(x)$. Also it is shown that an additive mapping h on R and a biadditive mapping $f: R \times R \rightarrow R$ satisfy $f(x, y)Rh(x) = 0$, for all $x, y \in R$, then $f(x, y)Rh(z) = 0$. Using these properties, we show that if R is a 2- divisible semiprime accessible ring, B is a biderivation and D is a derivation of R, then B and D are orthogonal if and only if one of the following equivalent conditions holds for every $x, y \in R$:

(i) $B(x, y)\, D(z) + D(x)\, B(z, y) = 0$
(ii) $D(x)\, B(x, y) = 0$ or $D(x)\, B(y, x) = 0$
(iii) $DB = 0$
(iv) DB is a biderivation

2. PRELIMINARIES

Kleinfled [3] studied the structure of the accessible rings.

A ring is defined to be accessible if the following two identities hold:

$$(x, y, z) + (z, x, y) - (x, z, y) = 0. \tag{1}$$

$$((w, x), y, z) = 0. \tag{2}$$

for all x, y, z in R, where the associator is defined as $(x, y, z) = (xy)z - x(yz)$ and the commutator is defined as

$[x, y] = xy - yx$ for all x, y, z in R.

Throughout this paper R will be an accessible ring. A ring R is said to be prime if $xRy = 0$ implies $x = 0$ or $y = 0$ and R is semiprime if $xRx = 0$ implies $x = 0$ for all $x, y \in R$. R is said to be 2- divisible if $2x = 0$, $x \in R$ implies $x = 0$.

Corresponding Author: P. S. Vijaya Lakshmi[1]*
[1,2]Department of Mathematics, Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

By substituting z = y in (1), we obtain the flexible law (y, x, y) = 0.

A linearization of this identity gives (y, x, z) = - (z, x, y).

Then equation (1) is (x, y, z) + (y, z, x) + (z, x, y) = 0.

In any arbitrary ring the identity

$$[xy, z] = x[y, z] + [x, z]y + (x, y, z) + (z, x, y) – (x, z, y) \text{ holds.}$$

From (1) this identity becomes [xy, z] = x[y, z] + [x, z]y.

Thus in an accessible ring R also we have the basic commutator identities

$$[x, yz] = [x, y]z + y[x, z] \text{ and } [xy, z] = x[y, z] + [x, z]y.$$

An additive mapping d: R→R is called a derivation if d(xy) = d(x)y + xd(y) for every x, y∈R. Two derivations f and g of an accessible ring R are called orthogonal if (f(x)R)g(y) = 0 = g(y) (Rf(x)) [1].

Following Daif. *et.al* [2], a biadditive map B: RxR→R is called a biderivation of R if B(xy, z) = B(x, z)y + xB(y, z) for all x, y, z∈R. A biderivation B and a Derivation D of R are called orthogonal if (B(x, y)R)D(z) = 0 = D(z)(RB(x, y)) for all x, y, z∈R.

We now consider some well-known results that will be needed in the subsequent results.

Lemma1: [[1], Lemma1] Let R be a 2- divisible semiprime ring and a, b∈R. Then the following are equivalent:

- axb = 0 for all x∈R
- bxa = 0 for all x∈R
- axb + bxa = 0 for all x∈R

If one of the above conditions is fulfilled, then ab = ba = 0, too.

Lemma 2: If R is a 2- divisible semiprime accessible ring, f and g are derivations of R such that (f(x)r) g(y) = 0 = g(y) (rf(x)), then f(x) (rg(y) = 0 = (g(y)r) f(x) for all x, y, r∈R.

Proof: Suppose f and g are such that (f(x)r) g(y) = 0 = g(y) (rf(x)) for all x, y, r∈R. (3)

Now we take g(y) = f(x) in (3), we have (f(x)r) f(x)=0.

Since R is flexible, f(x) (rf(x)) = 0.

Therefore f(x)rf(x) = 0. (4)

By linearizing (4) and using (3), we get

$$((f(x) + g(y))r)(f(x) + g(y)) = 0 = (f(x) + g(y))(r(f(x) + g(y))).$$

Now we expand ((f(x) + g(y))r) (f(x) + g(y)) = 0.

$$(f(x)r) f(x) + (f(x)r) g(y) + (g(y)r) f(x) + (g(y)r) g(y) = 0.$$

Using (4) and (3) in the above equation, we get (g(y)r) f(x) = 0. (5)

Again we expand ((f(x) + g(y)) (r(f(x) + g(y))) = 0

$$f(x) (rf(x)) + f(x) (rg(y)) + g(y) (rf(x) + g(y) (rg(y)) = 0.$$

Again using (4) and (3) in the above equation, we get f(x) (rg(y)) = 0. (6)

Therefore from (5) and (6), we have f(x) (rg(y)) = 0 = (g(y)r)f(x), for all x, y, r∈R.

This completes the proof of the lemma.

From this lemma, it follows that two derivations f and g of an accessible ring R are orthogonal if f(x)Rg(y) = g(y)Rf(x).

Lemma 3: Let R be a 2- divisible semiprime accessible ring. If an additive mapping h on R and a biadditive mapping f: $R\times R\to R$ satisfy $f(x, y)Rh(x) = 0$, for all $x, y\in R$, then $f(x, y)Rh(z) = 0$, for all $x, y, r\in R$.

Proof: Suppose f and h are such that $f(x, y)zh(x) = 0$, for all $x, y, z,\in R$. . (7)

Now we linearize $(x = x + s)$ the above equation, we get

$$f(x, y)zh(x) + f(x, y)zh(s) + f(s, y)zh(x) + f(s, y)zh(s) = 0, \text{ for all } x, y, z, s\in R. \quad (8)$$

From (7) and (8), we have $f(x, y)zh(s) + f(s, y)zh(x) = 0$

i.e., $f(x, y)zh(s) = -f(s, y)zh(x)$, for all $x, y, z, s\in R$.

By left multiplying this equation with $f(x, y)zh(s)$, we have

$$f(x, y)zh(s)Rf(x, y)\, zh(s) = -f(x, y)zh(s)Rf(s, y)\, zh(x) = 0, \text{ using (7).}$$

Since R is semiprime, the above relation implies $f(x, y)zh(s) = 0$.

This completes the proof of the lemma.

Lemma 4: Let D be a derivation and B a biderivation of a semiprime accessible ring R. The following identity holds for all $x, y, z\in R$: $DB(xy, z) = DB(x, z)y + D(x)B(y, z) + B(x, z)D(y) + x(DB)(y, z)$.

Proof: Let D and B are such that $DB(xy, z) = D(B(xy, z))$ for all $x, y, z\in R$. Now

$DB(xy, z) = D(xB(y, z) + B(x, z)y)$ for all $x, y, z\in R$, Then

$DB(xy, z) = D(x)B(y, z) + x(DB)(y, z) + (DB)(x, z)y + B(x, z)D(y)$.Thus

$DB(xy, z) = (DB)(x, z)y + D(x)B(y, z) + B(x, z)D(y) + x(DB)(y, z)$ for all $x, y, z\in R$.

The above lemmas are useful to prove the main results.

3. MAIN RESULTS

Now we prove the orthogonal conditions.

Theorem 1: Let R be a 2- divisible semiprime accessible ring. A biderivation B and a derivation D are orthogonal if and only if $B(x, y)D(z) + D(x)B(z, y) = 0$.

Proof: First we assume that B and D are such that

$$B(x, y)D(z) + D(x)B(z, y) = 0, \text{ for all } x, y, z\in R. \quad (9)$$

By taking $y = yr$ in (9), we get $B(x, yr)D(z) + D(x)B(z, yr) = 0$.

Then $(yB(x, r))\, D(z) + (B(x, y)r)\, D(z) + D(x)\, (yB(z, r)) + D(x)\, (B(z, y)r) = 0$.

Using (9) in the above equation, we have

$$(B(x, y)r)\, D(z) + D(x)\, (B(z, y)r) = 0, \text{ for all } x, y, z, r\in R. \quad (10)$$

Now we take $z = x$ in (10) and (9). Then

$$(B(x, y)r)\, D(x) + D(x)\, (B(x, y)r) = 0 \quad (11)$$

and $B(x, y)D(x) + D(x)B(x, y) = 0$.

We replace $D(x)$ by r in the last equation. So

$B(x, y)r + rB(x, y) = 0$.

$B(x, y)r = -rB(x, y)$.

By substituting this relation in (11), we obtain

$$(B(x, y)r)D(x) - D(x)(rB(x, y)) = 0 \text{ for all } x, y, r\in R. \quad (12)$$

Then $D(x)\, (RB(x, y)) = (0)$, according to lemma 1.

Hence by lemma 3, we get $D(x)\, (RB(z, y)) =0$.

In particular, $D(x)\, (RB(z, y)) = (0) = (B(z, y)R)\, D(x)$, for all $x, y, z\in R$ by lemma 1.

Therefore by lemma 2, we have $D(x)RB(z, y) = (0) = B(z, y)RD(x)$.

So B and D are orthogonal.

Conversely if B and D are orthogonal, then $D(x)\, B(z, y) = (0) = B(x, y)\, D(z)$, by lemma 3.

Thus $B(x, y)D(z) + D(x)B(z, y) = 0$.

This completes the proof of the theorem.

Theorem 2: Let R be a 2- divisible semiprime accessible ring. A biderivation B and a derivation D are orthogonal if and only if $D(x)B(x, y) = 0$ or $D(x)B(y, x) = 0$ for all $x, y \in R$.

Proof: We assume B and D are such that $D(x)B(x, y) = 0$, for all $x, y \in R$. (13)

Also $(D(x)B(x, y))s = 0 = s(D(x)B(x, y))$, for all $x, y, s \in R$.

Thus $D(x)B(x, y)s = 0$. (14)

By substituting $y = ys$ in (13), we get

$$D(x)yB(x, s) + D(x)B(x, y)s = 0.$$

From (14), we have $D(x)yB(x, s) = 0$.

By lemma 3, $D(z)RB(x, s) = 0$.

By multiplying this with $B(x, s)w$ on left and $wD(z)$ on right, we have

$$B(x, s)wD(z)RB(x, s)wD(z)= 0.$$

Since R is semiprime, $B(x, s)wD(z) = 0$.
i.e., $B(x, s)RD(z) = 0$.

Hence B and D are orthogonal.

Similarly, we can prove that if $D(x)B(y, x) = 0$, then D and B are orthogonal.

Conversely, if D and B are orthogonal, then $D(x)RB(x, y) = (0)$ for all $x, y \in R$.

Therefore $D(x)B(x, y) = 0$, according to lemma 1.

Similarly $D(x)B(y, x) = 0$.

This completes the proof of the theorem.

Theorem 3: Let R be a 2- divisible semiprime accessible ring. A biderivation B and a derivation D are orthogonal if and only if $DB = 0$.

Proof: We assume B and D are such that $DB = 0$. (15)

By lemma 4, we have

$$DB(xy, z) = (DB)(x, z)y + D(x)B(y, z) + B(x, z)D(y) + x(DB)(y, z),$$

Using (15), we get $D(x)B(y, z) + B(x, z)D(y) = 0$ for all $x, y, z \in R$.

By interchanging y and z, we have

$$D(x)B(z, y) + B(x, y)D(z) = 0.$$

Hence by theorem 1, D and B are orthogonal.

Conversely if D and B are orthogonal then $D(x)sB(y, z) = 0$ for all $x, y, z, s \in R$.

Hence $\quad 0 = D(D(x)sB(y, z))$

$0 = D(D(x))sB(y, z) + D(x)D(sB(y, z))$.
$0 = D(D(x))sB(y, z) + D(x)D(s)B(y, z) + D(x)s(DB)(y, z)$.

The sum of the first two terms is zero as D and B are orthogonal. So we have
$D(x)\, s\, (DB)(y, z) = 0$, for all $x, y, z, s \in R$.

Let $x=B(y, z)$ and we substitute in the above equation.

Then we get $(DB)(y, z)R(DB)(y, z) = (0)$ for all $y, z \in R$.

Since R is semiprime, $(DB)(y, z) = 0$ for all $y, z \in R$.

Hence $DB=0$.

This completes the proof of theorem.

Theorem 4: Let R be a 2- divisible semiprime accessible ring. Then a biderivation B and a derivation D are orthogonal if and only if DB is a biderivation.

Proof: Let B and D be such that DB is a biderivation.

Then $DB(xy, z)= (DB)(x, z)y + x(DB)(y, z)$, for all $x, y, z \in R$. (16)

But by lemma 4, we have
$DB(xy, z) = DB(x, z)y + D(x)B(y, z) + B(x, z)D(y) + x(DB)(y, z)$ for all $x, y, z \in R$. (17)

From (16) and (17), we get $D(x)B(y, z) + B(x, z)D(y) = 0$, for all $x, y, z \in R$.

So by the proof of the first part of theorem 1, we have that D and B are orthogonal.

Conversely, let D and B be orthogonal. Then theorem 1 implies that
$D(x)B(y, z) + B(x, z)D(y) = 0$, for all $x, y, z \in R$. (18)

Again by lemma 4, we get $DB(xy, z) = (DB)(x, z)y + x(DB)(y, z)$, for all $x, y, z \in R$.

It is clear now that DB is a biderviation. This completes the proof of the theorem.

Theorem 5: Let R be a 2- divisible semiprime accessible ring, B be a biderivation and D be a derivation on R. Then B and D are orthogonal if and only if the following conditions are equivalent:
(i) $B(x, y)D(z) + D(x)B(z, y) = 0$, for every $x, y \in R$.
(ii) $D(x)B(x, y) = 0$ or $D(x)B(y, x) = 0$, for every $x, y \in R$.
(iii) $DB = 0$
(iv) DB is a biderivation

Proof: It follows easily from theorems 1,2,3,4.

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