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EVEN SQUARE ALGEBRAIC STRUCTURES

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ABSTRACT

In this note we introduce the notion of even square algebraic structures. Even square algebraic structures make non-explicit appearances in the mathematical literatures and the idea to introduce them explicitly has originated through author's recent work on even square rings and subsequently even square semigroups.

Key-Words: *Even square algebraic structure, even square ring, even square semigroup, even square element, nil element, nilpotent element, zero divisor.*

MSC 2010: *08A99.*

1. INTRODUCTION

Even square rings [1] and semigroups [2] have been introduced recently. The idea to introduce the notion of even square algebraic structures has originated from [1, 2]. In the mathematical literatures [3-6] even square algebraic structures have implicit presence however they do have some unique properties.

In this note an algebraic structure is equipped with two binary operations: addition and multiplication or with one multiplicative binary operation and a unary operation defined by $f(a) = 2a, \forall a \in A$.

Let A be an algebraic structure. An element $a \in A$ is called an even element of A if $a \in 2A$. An element a of A is called an even square element if $a^2 \in 2A$. A is called an even square algebraic structure if $a^2 \in 2A, \forall a \in A$. In other words an algebraic structure A is called an even square algebraic structure if every element of A is an even square element.

Every even element of an algebraic structure is an even square element however an even square element is not necessarily an even element. Let a and b are any two even square elements of A then ab is an even square element if A is commutative.

Let A be any algebraic structure then the set of all even elements of A forms a substructure of A and the set of all even square elements forms a substructure of A if A is commutative.

It is worth to note that nil elements (which are special type of nilpotent elements) have special role in finite even square algebraic structures.

2. EVEN SQUARE PROPERTY

An algebraic structure A is said to have even square property if $a^2 \in 2A, \forall a \in A$. Therefore every even square algebraic structure has even square property. It may be noted that a non-empty set N is an even square set iff it possesses even square property [2].

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Each element of an even square algebraic structure is an even square element however each element of a substructure of an even square algebraic structure is not necessarily an even square element.

It has been noted in [2] that the even square property is not a hereditary property of semigroups. We see that the even square property is not a hereditary property of groups. Let $G = \{2,4\}$. Then G is an even square group under multiplication modulo 6. Clearly $S = \{4\}$ is a non-even square subgroup of G under multiplication modulo 6.

If we consider the ring R of real numbers then Z is a subring of R and Z does not have even square property. Similarly one may easily see that even square property is not a hereditary property of modules and it is also clear that every integral domain or ideal contained in an even square ring is not necessarily an even square integral domain or ideal. However every field of characteristic $\neq 2$ possesses even square property. Even square property is a hereditary property of fields as each subfield of an even square field is an even square field. The multiplicative group of all nonzero elements of an even square field has even square property.

3. NIL ELEMENTS

Let A be an algebraic structure with a multiplicative binary operation. Then an element $a \in A$ is called a nil element of A if $a^2 = 2a = 0$. In this case $2a$ is the result of the unary operation defined on A. Let A be an algebraic structure with two binary operations addition and multiplication then an element $a \in A$ is called a nil element of A if $a^2 = a + a = 2a = 0$. Here $2a$ is the result of the additive binary operation defined on A.

Let b is any element of A and a is a nil element of A then $(ab)^2 = 2ab = 0$ provided A is commutative. Therefore if A is commutative and a is a nil element then ab is nil for all $b \in A$.

Let A be a commutative even square algebraic structure and a be a unique nonzero nil element of A then we can easily see that $ab = ba = 0$, $\forall b \in A$. Thus a annihilates A. However a unique non-zero nil element of a non-even square algebraic structure A does not necessarily annihilate A. For example, 2 is a nil element of Z_4 but 2 does not annihilate Z_4.

It may be noted that each nilpotent element of index two is not necessarily a nil element. We can see this easily. Let $x, y \in A$ such that $x^2 = 2x = 0$, $y^2 = 0,\ 2y = x, xy = 0$. Then y is a non-nil nilpotent element of index two. It is also notable that if A does not have unique non-zero nil element then each nil element of A does not necessarily annihilates A. For example let $x, y, z \in A$ such that $x^2 = 2x = 0,\quad y^2 = 2y = 0,\quad z^2 = 2z = 0.$ and $xy = yx = z, xz = zx = 0,\quad yz = zy = 0$. Clearly x does not annihilate A.

It may be noted that the set of all nil elements in a commutative ring forms an ideal of the ring. The nil elements determine the minimum number of ideals in commutative as well as non- commutative even square rings. An even square ring (commutative as well as non commutative) containing $n > 2$ nil elements has at least $n+2$ ideals provided each nil element annihilates the ring. If the order of the ring is n and it contains n nil elements then it is a commutative even square ring and has $n+1$ ideals provided $n > 2$ and each nil element annihilates the ring [1].

4. ZERO DIVISORS AND NILPOTENT ELEMENTS

Even square algebraic structures of finite order are good source of examples of algebraic structures with zero divisors and nilpotent elements.

We have seen in the above section (3) that if an even square algebraic structure A contains a unique non-zero nil element a then it annihilates A and hence each non-zero element of A is a zero divisor. Similarly there are commutative as well as non-commutative even square algebraic structures A containing more than two nil elements such that each nil element annihilates A and clearly in this case also each element of A is a zero divisor. Let A is an even square ring of order 2^n, $n \in Z^+$ then each element of A is nilpotent. Clearly this result holds for non-commutative even square ring as well.

CONCLUDING REMARKS

Among several even square algebraic structures even square semigroups and even square rings are of particular importance. The notion of nil elements naturally arises in even square semigroups and rings however their existence in non-even square algebraic structures is also notable and has meaningful implications. Even square algebraic structure also exists as a substructure of a non-even square algebraic structure and plays interesting role.

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