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COMMON FIXED POINT THEOREM WITH INTIMATE MAPPINGS IN DISLOCATED METRIC SPACE

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ABSTRACT

The purpose of the paper is to generalize common fixed point theorem of Jain and Bajaj [4] in dislocated metric space using intimate mappings.

***Keywords**: Dislocated metric space, intimate mappings.*

1. INTRODUCTION

Banach's common fixed point theorem was generalized in 1976 by Jungck by the use of commuting maps. The result has been since generalized and extended in various ways by many authors. Jungck [2] introduced the concept of compatibility which is a generalization of weak commutativity. The concept of compatible mapping of type (A) has further been generalized through the concept of intimate mappings.

2. PRELIMINARIES

Definition 2.1 [1]: Let X be a non-empty set and let d: X×X→ [0, ∞) be a function satisfying the following conditions:

(i) d(x, y) = d(y, x)
(ii) d(x, y) = d(y, x) = 0 implies x = y
(iii) d(x, y) ≤ d(x, z) +d(z, y) for all x, y, z ϵX

Then d is called dislocated metric (or simply d- metric) on X.

Definition 2.2: Let A and S be self-maps on a d-metric space X. The pair {A, S} is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ =t for some t ϵ X.

Definition 2.3: Let A and S be self-maps on a d-metric space X. The pair {A, S} is said to be compatible of type (A) if $\mathrm{Lim}_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\mathrm{Lim}_{n\to\infty} d(ASx_n, AAx_n) = 0$ whenever $\{x_n]$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ = t for some t ϵ X.

Definition 2.4: Let A and S be self-maps on a d-metric space X. The pair {A, S} is said to be S intimate iff $\alpha d(SAx_n, Sx_n) \le \alpha d(AAx_n, Ax_n)$ where α= lim sup or lim in f whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ =t for some t ϵ X.

MAIN RESULT

Let A, B, E, F, S, T be six mappings on dislocated metric space (X, d) such that

2.1.1 A(X) ⊂ F(X)∪S(X) and B(X) ⊂ E(X) ∪T(X)

2.1.2 $$d(Ax, By) \le a_1 d(Fy, By)\frac{[1+d(Ex,Ax)]}{[1+d(Tx,Sy)]} + a_2 [d(Tx, Ax)+ d(Sy, By)] + a_3 d(Ex, Ax)\frac{[1+d(Fy,By)]}{[1+d(Sy,By)]} + a_4 [d(Ex, Fy)+ d(Fy, Ax)] + a_5 [d(Tx, By) + d(Ax, Sy)] + a_6 d(Fy, By)\frac{[1+d(Ax,Tx)]}{[1+d(Sy,Ex)]}$$

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For all x, y in X where $a_1, a_2, a_3, a_4, a_5, a_6 \geq 0$ and $a_1+2a_2+a_3+2a_4+2a_5+a_6 < \frac{1}{2}$

2.1.3 The pair {A, E} is E intimate and {B, F} is S intimate.

2.1.4 E(X) and T(X) are complete.

Then A, B, E, F, S, T have a unique common fixed point.

Proof: Let x_0 be any arbitrary point in X. Then from 2.1.1 there exists a point x_1 in X such that $Ax_0=Fx_1=Sx_1$. For point x_1 we choose a point $x_2 \in X$ such that $Bx_1=Ex_2=Tx_2$ and so on .Inductively we define a sequence $\{y_n\}$ in X such that $y_{2n}=Fx_{2n+1}=Sx_{2n+1}= Ax_{2n}$ and $y_{2n+1}=Ex_{2n+2}=Tx_{2n+2}=Bx_{2n+1}$ for n=0, 1, 2.....

We now prove $\{y_n\}$ is a Cauchy sequence in X.

Consider

$$
\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Ax_{2n},Bx_{2n+1}) \\
&\leq a_1\, d(Fx_{2n+1},Bx_{2n+1})\frac{[1+d(Ex_{2n},Ax_{2n})]}{[1+d(Tx_{2n},Sx_{2n+1})]} + a_2\,[d(Tx_{2n},Ax_{2n})+ d(Sx_{2n+1},B_{2n+1})] \\
&\quad + a_3\, d(Ex_{2n},Ax_{2n})\frac{[1+d(Fx_{2n+1},Bx_{2n+1})]}{[1+d(Sx_{2n+1},Bx_{2n+1})]} + a_4\,[d(Ex_{2n},Fx_{2n+1})+ d(Fx_{2n+1},Ax_{2n})] \\
&\quad + a_5\,[d(Tx_{2n},Bx_{2n+1}) + d(Ax_{2n},Sx_{2n+1})] + a_6\, d(Fx_{2n+1},Bx_{2n+1})\frac{[1+d(Ax_{2n},Tx_{2n})]}{[1+d(Sx_{2n+1},Ex_{2n})]} \\
&\leq a_1 d(y_{2n},y_{2n+1})[\frac{[1+d(y_{2n},y_{2n+1})]}{[1+d(y_{2n-1},y_{2n})]} + a_2[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] + a_3 d(y_{2n-1},y_{2n}) \frac{[1+d(y_{2n},y_{2n+1})]}{[1+d(y_{2n},y_{2n+1})]} \\
&\quad + a_4[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n})] + a_5[d(y_{2n-1},y_{2n+1}) + d(y_{2n},y_{2n})] + a_6 d(y_{2n},y_{2n+1}) \frac{[1+d(y_{2n},y_{2n-1})]}{[1+d(y_{2n},y_{2n-1})]} \\
&\leq a_1 d(y_{2n},y_{2n+1}) + a_2[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] + a_3 d(y_{2n-1},y_{2n}) + a_4[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n})] \\
&\quad + a_5\,[d(y_{2n-1},y_{2n+1}) + d(y_{2n},y_{2n})] + a_6 d(y_{2n},y_{2n+1}) \\
&\leq a_1 d(y_{2n},y_{2n+1}) + a_2[d(y_{2n-1}, y_{2n}) + d(y_{2n},y_{2n+1})] + a_3 d(y_{2n-1},y_{2n}) + a_4[d(y_{2n-1},y_{2n}) + d(y_{2n}, y_{2n-1}) \\
&\quad + d(y_{2n-1}, y_{2n})] + a_5[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n+1})] + a_6 d(y_{2n}, y_{2n+1}) \\
&\leq (a_1+a_2+3a_5+a_6)d(y_{2n}, y_{2n+1}) + (a_2+a_3+3a_4+a_5)d(y_{2n-1},y_{2n})
\end{aligned}
$$

$$d(y_{2n},y_{2n+1}) \leq \frac{a_2+a_3+3a_4+a_5}{(1-(a_1+a_2+3a_5+a_6)} d(y_{2n-1},y_{2n})$$

Let $\frac{a_2+a_3+3a_4+a_5}{(1-(a_1+a_2+3a_5+a_6)} = h$

Therefore

$$d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n)$$

$d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n) \leq .... \leq h^n d(y_0, y_1)$ for every integer p > 0 we get

$$
\begin{aligned}
d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1})+ d(y_{n+1}, y_{n+2})+ d(y_{n+3}, y_{n+4})+\ldots\ \ldots\ \ldots+ d(y_n, y_{n+p}) \\
&\leq h^n d(y_0, y_1)+h^{n+1} d(y_0, y_1)+h^{n+2} d(y_0, y_1) +\ldots\ldots + h^{n+p-1} d(y_0, y_1) \\
&\leq h^n[1+h+h^2+h^3\ldots\ \ldots\ h^{p-1}]d(y_0, y_1) \\
&\leq \frac{h^n}{1-h} d(y_0, y_1)
\end{aligned}
$$

As $n\to\infty$ we get $d(y_n, y_{n+p}) \to 0$

Therefore $\{y_n\}$ is a Cauchy sequence

Since E(X) and T(X) are complete and $\{Ex_{2n}\}$ and $\{Tx_{2n}\}$ is Cauchy therefore it converges to a point z = Eu =Tu for some u in X. Then $y_n \to z$ and $Ax_{2n}, Bx_{2n+1}, Ex_{2n}, Fx_{2n+1}, S_{2n+1}, Tx_{2n} \to z$

From from 2.1.2

$$
\begin{aligned}
d(Au, Bx_{2n+1}) &\leq a_1\, d(Fx_{2n+1}, Bx_{2n+1})\frac{[1+d(Eu,Au)]}{[1+d(Tu,Sx_{2n+1})]} + a_2\,[d(Tu, Au) + d(Sx_{2n+1}, B_{2n+1})] \\
&\quad + a_3 d(Eu, Au)\frac{[1+d(Fx_{2n+1},Bx_{2n+1})]}{[1+d(Sx_{2n+1},Bx_{2n+1})]} + a_4\,[d(Eu, Fx_{2n+1}) + d(Fx_{2n+1}, Au)] \\
&\quad + a_5\,[d(Tu, Bx_{2n+1}) + d(Au, Sx_{2n+1})] + a_6\, d(Fx_{2n+1}, Bx_{2n+1})\frac{[1+d(Au,Tu)]}{[1+d(Sx_{2n+1},Eu)]} \\
&\leq a_1 d(z, z) \frac{[1+d(z,Au)]}{[1+d(z,z)]} + a_2[d(z, Au)+d(z, z)] + a_3 d(z, Au) \frac{[1+d(z,z)]}{[1+d(z,z)]} + a_4[d(z, z)+d(z, Au)] \\
&\quad + a_5[d(z, z)+d(Au, z)] + a_6 d(z, z) \frac{[1+d(Au,z)]}{[1+d(z,z)]} \\
&\leq a_1 d(z, Au) + a_2[d(z, Au) + d(z, Au) + d(z, Au)] + a_3 d(z, Au) + a_4[d(z, Au) + d(z, Au) + d(z, Au)] \\
&\quad + a_5[d(z, Au)+d(Au, z) + d(z, Au)]+ a_6 d(z, Au)\, d(z, Au) \\
&\leq (a_1 + 3a_2 +a_3 +3a_4+3a_5+a_6)d(z, Au)
\end{aligned}
$$

$[1- (a_1 + 3a_2 + a_3 + 3a_4 + 3a_5 + a_6)]d(z, Au) \leq 0$

This implies d(z, Au) =0 = d(Au, z). Therefore Au = z.

Since $A(X) \subset F(X) \cup S(X)$ there exists $x \in X$ such that Fw = Sw = z

Then consider

$$d(z, Bw) = d(Au, Bw)$$
$$\leq a_1 d(Fw, Bw)\frac{[1+d(Eu, Au)]}{[1+d(Tu, Sw)]} + a_2 [d(Tu, Au) + d(Sw, Bw)] + a_3 d(Eu, Au)\frac{[1+d(Fw, Bw)]}{[1+d(Sw, Bw)]}$$
$$+ a_4 [d(Eu, Fw) + d(Fw, Au)] + a_5 [d(Tu, Bw) + d(Au, Sw)] + a_6 d(Fw, Bw)\frac{[1+d(Au, Tu)]}{[1+d(Sw, Eu)]}$$
$$\leq a_1 d(z, Bw)\frac{[1+d(z,z)]}{[1+d(z,z)]} + a_2[d(z, z)+d(z, Bw)] + a_3 d(z, z)\frac{[1+d(z,Bw)]}{[1+d(z,Bw)]} + a_4[d(z, z)+d(z, z)] + a_5[d(z, Bw)+d(z, z)]$$
$$+ a_6 d(z, Bw)\frac{[1+d(z,z)]}{[1+d(z,z)]}$$
$$\leq (a_1 + 3a_2 + 2a_3 + 4a_4 + 3a_5 + a_6)d(z, Bw)$$

$[1- (a_1 + 3a_2 + 2a_3 + 4a_4 + 3a_5 + a_6)]d(z, Bw) \leq 0$

This implies d(z, Bw)=0 .Similarly d(Bw, z)=0 which implies Bw = z

Since Au=Eu=Tu=z

As the pair {A, E} is E intimate we have

$$d(EAx_{2n}, Ex_{2n}) \leq d(AAx_{2n}, Ax_{2n})$$

$$d(Ez, z) \leq d(Az, z)$$

Now we prove Az=z

$$d(Az, z) = d(Az, Bw)$$
$$\leq a_1 d(Fw, Bw)\frac{[1+d(Ez, Az)]}{[1+d(Tz, Sw)]} + a_2 [d(Tz, Az) + d(Sw, Bw)] + a_3 d(Ez, Az)\frac{[1+d(Fw, Bw)]}{[1+d(Sw, Bw)]}$$
$$+ a_4 [d(Ez, Fw) + d(Fw, Az)] + a_5 [d(Tz, Bw) + d(Az, Sw)] + a_6 d(Fw, Bw)\frac{[1+d(Az, Tz)]}{[1+d(Sw, Ez)]}$$
$$\leq a_1 d(z, z)\frac{[1+d(Ez,Az)]}{[1+d(Tz,z)]} + a_2[d(Tz, Az)+d(z, z)] + a_3 d(Ez, Az)\frac{[1+d(z,z)]}{[1+d(z,z)]} + a_4[d(Ez, z)+d(z, Az)]$$
$$+ a_5[d(Tz, z)+d(Az, z)] + a_6 d(z, z)\frac{[1+d(Az,Tz)]}{[1+d(z,Ez)]}$$
$$\leq a_1 d(z, z)\frac{[1+d(Az,Az)]}{[1+d(Az,z)]} + a_2[d(Az, Az)+d(z, z)] + a_3 d(Az, Az)\frac{[1+d(z,z)]}{[1+d(z,z)]} + a_4[d(Az, z)+d(z, Az)]$$
$$+ a_5[d(Az, z)+d(Az, z)] + a_6 d(z, z)\frac{[1+d(Az,Az)]}{[1+d(z,Az)]}$$

$[1- (2a_1 + 4a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6)]d(Az, z) \leq 0$

d(Az, z) = 0.

Similarly d(z, Az) = 0. Therefore Az = z. This implies Az = Tz = Ez = z. Similarly the pair {B, F} is F intimate we have

$$d(FBx_{2n+1}, Fx_{2n+1}) \leq d(BBx_{2n+1}, Bx_{2n+1})$$

Taking limits $n \to \infty$.

$d(Fp, p) \leq d(Bp, p)$. Now we prove BZ=z.

$$d(z, Bz) = d(Az, Bz)$$
$$\leq a_1 d(Fz, Bz)\frac{[1+d(Ez, Az)]}{[1+d(Tz, Sz)]} + a_2 [d(Tz, Az) + d(Sz, Bz)] + a_3 d(Ez, Az)\frac{[1+d(Fz, Bz)]}{[1+d(Sz, Bz)]}$$
$$+ a_4 [d(Ez, Fz) + d(Fz, Az)] + a_5 [d(Tz, Bz) + d(Az, Sz)] + a_6 d(Fz, Bz)\frac{[1+d(Az, Tz)]}{[1+d(Sz, Ez)]}$$
$$\leq a_1 d(Fz, Bz)\frac{[1+d(z, z)]}{[1+d(z, Sz)]} + a_2 [d(z, z) + d(Sz, Bz)] + a_3 d(z, z)\frac{[1+d(Fz, Bz)]}{[1+d(Sz, Bz)]}$$
$$+ a_4 [d(z, Fz) + d(Fz, z)] + a_5 [d(z, Bz) + d(z, Sz)] + a_6 d(Fz, Bz)\frac{[1+d(z, z)]}{[1+d(Sz, z)]}$$
$$\leq a_1 d(Bz, Bz)\frac{[1+d(z, z)]}{[1+d(z, Bz)]} + a_2 [d(z, z) + d(Bz, z)] + a_3 d(z, z)\frac{[1+d(Bz, Bz)]}{[1+d(Bz, Bz)]} + a_4 [d(z, Bz) + d(Bz, z)] + a_5 [d(z, Bz)$$
$$+ d(z, Bz)] + a_6 d(Bz, Bz)\frac{[1+d(z, z)]}{[1+d(z, Bz)]}$$
$$\leq 2a_1 + 4a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6)d(z, Bz)$$

$[1- (2a_1 +4a_2 +2a_3 +2a_4+2a_5+2a_6)]d(z, Bz) \leq 0$

d(z, Bz)=0.

Similarly d(Bz, z) = 0. Therefore Bz = z. This implies Bz = Fz = Sz = z. Therefore z is the common fixed point of A, B, E, F, S, T. Now we prove z is the unique fixed point of A, B, E, F, S, T

Let p is the another common fixed point of A, B, E, F, S, T.

$$d(z, p) = d(Az, Bp)$$
$$\leq a_1 d(Fp, Bp)\frac{[1+d(Ez,Az)]}{[1+d(Tz,Sp)]} + a_2 [d(Tz, Az) + d(Sp, Bp)] + a_3 d(Ez, Az)\frac{[1+d(Fp,,Bp)]}{[1+d(Sp,Bp)]}$$
$$+ a_4 [d(Ez, Fp) + d(Fp, Az)] + a_5 [d(Tz, Fp) + d(Az, Sp)] + a_6 d(Fp, Bp)\frac{[1+d(Az,Tz)]}{[1+d(Sp,Ez)]}$$
$$\leq a_1 d(p, p)\frac{[1+d(z,z)]}{[1+d(z,p)]} + a_2 [d(z, z)+ d(p,p)] + a_3 d(z, z)\frac{[1+d(p,,p)]}{[1+d(p,p)]} + a_4 [d(z, p)+ d(p, z)]$$
$$+ a_5 [d(z, p) + d(z, p)] + a_6 d(p, p)\frac{[1+d(z,z)]}{[1+d(p,z)]}$$
$$\leq 2a_1 +4a_2 +2a_3 +2a_4+2a_5+2a_6)d(z, p)$$

$[1- (2a_1 +4a_2 +2a_3 +2a_4+2a_5+2a_6)]d(z, p) \leq 0$

d(z, p) = 0. Similarly d(p, z)=0. Therefore p = z. Hence z is the unique common fixed point of A, B, E, F, S, T.

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