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COMPARISON THEOREMS FOR SUMMATION EQUATIONS

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ABSTRACT

This paper is devoted to obtain comparison results for solution of summation equation.

$$x(t) = x_0 + \sum_{s=t_0}^{t-1} K(t, s, x(s)).$$

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1. INTRODUCTION:

Agarwal [1], Kelley and Peterson [9] developed the theory of difference equations and difference inequalities. Some comparison results for difference equations are obtained by K.L. Bondar [2, 3], V. Kabada, Otero-Espianar [7] and P. Eloe [8]. Some summation inequalities are discussed by K.L. Bondar [4, 5]. Comparison results for nonlinear difference equations using maximal and minimal solutions are obtained by K.L. Bondar, V.C. Borkar, S.T. Patil [6]. Some differential and integral inequalities are given in [10].

In this paper, we obtain some comparison results of solution of the summation equation

$$x(t) = x_0 + \sum_{s=t_0}^{t-1} K(t, s, x(s)). \tag{1}$$

2. PRELIMINARY NOTES:

Let $J = \{t_0, t_0 + 1\ldots t_0 + a\}$, $t_0 \geq 0$, $t_0 \in R$, and E be an open subset of R. Consider the difference equations with an initial condition,

$$\Delta u(t) = g(t, u(t)),\ u(t_0) = u_0 \tag{2}$$

where $u_0 \in E$, $u: J \to E$, $g : J \times E \to R$.

Definition: 2.1 *The function $\phi : J \to R$ is said to be a solution of initial value problem (2), if it satisfies $\Delta\phi(t) = g(t, \phi(t))$; $\phi(t_0) = u_0$.*

The initial value problem (2) is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convention $\sum_{s=t_0}^{t_0-1} g(s, u(s)) = 0$ and so $u(t)$ given above is the solution of (2).

Definition: 2.2 Let $r(t)$ be any solution of (1) on J. Then is said to be maximal solution of (1), if every solution of $x(t)$ of (1) existing on J, the inequality $x(t) \leq r(t)$ holds for $t \in J$.

Definition: 2.2 A solution $\rho(t)$ of (1) is said to be minimal solution of (1), $\rho(t) \leq x(t)$ holds for $t \in J$. Author proved following theorem in [5].

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Theorem: 2.2[5] Assume *that*

(i) K: J × J × R → R and K (t, s, x) is nondecreasing in x for each fixed (t, s) and one of the inequalities

$$x(t) \le h(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)),$$

$$y(t) \ge h(t) + \sum_{s=t_0}^{t-1} K(t, s, y(s))$$

is strict where x,y : J → R ;

(ii) $x(t_0) < y(t_0)$.

Then

$$x(t) < y(t), \quad t \ge t_0.$$

3. COMPARISON RESULTS:

In this section we obtain the comparison results on solution of (1).

Theorem: 3.1 Let $G: J \times J \times R_+ \to R_+$ is continuous, $G(t, s, u)$ is monotone nondecreeasing in u for each (t, s) and

$$m(t) \le m_0(t) + \sum_{s=t_0}^{t-1} G(t, s, m(s)), \quad t \ge t_0,$$

where m:J→ R is continuous. Suppose that r(t) is the maximal solution of the summation equation

$$u(t) = u_0(t) + \sum_{s=t_0}^{t-1} G(t, s, u(s)) \tag{3}$$

existing on J. Then the inequality $m(t_0) \le u_0(t_0)$ *implies*

$$m(t) \le r(t), \qquad t \ge t_0. \tag{4}$$

Proof: Let $u(t, \epsilon)$ be any solution of summation equation

$$u(t) = u_0(t) + \epsilon + \sum_{s=t_0}^{t-1} G(t, s, u(s))$$

for $\epsilon > 0$ sufficiently small. Since

$$\lim_{\epsilon \to 0} x(t, \epsilon) \equiv r(t),$$

it is enough to show that

$$m(t) < u(t, \epsilon), \qquad t \ge t_0.$$

Observe that $m(t_0) < u(t_0, \epsilon)$ and $u(t, \in) > u_0(t) + \sum_{s=t_0}^{t-1} G(t, s, u(t, s, \in))$.

Hence an application of Theorem 2.4 shows that the inequality (5) is valid. This establishes the theorem.

We shall prove an extension of the result of Theorem 3.1 to systems of summation inequalities. The proof of that will be presented using partial ordering in R^n .

Let us introduce the relation ≤ in R^n , namely, we set, for any two elements $x, y \in R^n$,

$x \le y$ if and only if $x_i \le y_i$ for each $i=1,2,\ldots,n$. (6)

This relation induces a partial ordering in R^n and it is easy that, for any bounded set $A \subset R^n$, there exists the sup A with respect to the relation (6), which is

$$\sup A = \min\,[z \in R^n: x \le z \text{ for each } x \in A]. \tag{7}$$

In fact, we need (7) only for two elements sets, in which case we have

$$sup\ [x, y] = z = (z_1, z_2, \dots, z_n), \tag{8}$$

where $z_i = max(x_i, y_i)$, x_i, y_i being the components of x and y, respectively. We are now in a position to prove the following result.

Theorem: 3.2 *Let $K: J \times J \times R^n \to R^n$ is continuous, $K(t, s, x)$ is monotonic nondecreasing in x for each (t, s) and*

$$x(t) \le x_0(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)), \tag{9}$$

where $x, x_0 : J \to R^n$ *is continuous. Suppose that r (t) is the maximal solution of the summation equation*

$$u(t) = x_0(t) + \sum_{s=t_0}^{t-1} K(t, s, u(s)) \tag{10}$$

existing on J. Then

$$x(t) \le r(t), \qquad t \ge t_0. \tag{11}$$

Proof: Define

$$F(t, s, y) = K(t, s, sup[y, x(t)]). \tag{12}$$

By (8), $x(t) \le sup[y, x(t)]$ and therefore it follows, from the monotonicity of K and (12), that

$$F(t, s, y) \ge K(t, s, x(t)) \text{ for each } y. \tag{13}$$

Let $r^*(t)$ be a maximum solution of

$$u(t) = x_0(t) + \sum_{s=t_0}^{t-1} F(t, s, u(s))$$

existing on J. Then using (13) and (9), we get

$$r^*(t) = x_0(t) + \sum_{s=t_0}^{t-1} F(t, s, r^*(s))$$

$$\ge x_0(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)).$$

Hence

$$r^*(t) \ge x(t). \tag{14}$$

It then results from (14) and (8) that

$$sup\ [r^*(t), x(t)] = K(t, s, r^*(t)),$$

and consequently, by (12),

$$F(t, s, r^*(t)) = K(t, s, r^*(t)]).$$

Thus r*(t) is also the maximal solution of (10). Hence (14) proves the desired result (11). The proof is complete.

Corollary: 3.3 *Let $f: J \times R^n \to R^n$ is continuous, (t, x) is monotonic nondecreasing in x for each t and*

$$x(t) \le x_0(t) + \sum_{s=t_0}^{t-1} f(s, x(s)),$$

where $x : J \to R^n$ *is continuous. Suppose that r(t) is the maximal solution of*

$$\Delta y(t) = f(t, y), \quad y(t_0) = x_0,$$

existing on J. Then

$$x(t) \le r(t), \qquad t \ge t_0.$$

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