# COMPARISON THEOREMS FOR SUMMATION EQUATIONS 

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ABSTRACT
This paper is devoted to obtain comparison results for solution of summation equation.

$$
x(t)=x_{0}+\sum_{s=t_{0}}^{t-1} K(t, s, x(s))
$$

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## 1. INTRODUCTION:

Agarwal [1], Kelley and Peterson [9] developed the theory of difference equations and difference inequalities. Some comparison results for difference equations are obtained by K.L. Bondar [2, 3], V. Kabada, Otero-Espianar [7] and P. Eloe [8]. Some summation inequalities are discussed by K.L. Bondar [4, 5]. Comparison results for nonlinear difference equations using maximal and minimal solutions are obtained by K.L. Bondar, V.C. Borkar, S.T. Patil [6]. Some differential and integral inequalities are given in [10].

In this paper, we obtain some comparison results of solution of the summation equation

$$
\begin{equation*}
x(t)=x_{0}+\sum_{s=t_{0}}^{t-1} K(t, s, x(s)) \tag{1}
\end{equation*}
$$

## 2. PRELIMINARY NOTES:

Let $J=\left\{t_{0}, t_{0}+1 \ldots t_{0}+a\right\}, t_{0} \geq 0, t_{0} \in R$, and $E$ be an open subset of $R$. Consider the difference equations with an initial condition,

$$
\begin{equation*}
\Delta u(t)=g(t, u(t)), u\left(t_{0}\right)=u_{0} \tag{2}
\end{equation*}
$$

where $u_{0} \in E, u: J \rightarrow E, g: J \times E \rightarrow R$.
Definition: 2.1 The function $\phi: J \rightarrow R$ is said to be a solution of initial value problem (2), if it satisfies $\Delta \phi(t)=g(t, \phi(t)) ; \quad \phi\left(t_{0}\right)=u_{0}$.

The initial value problem (2) is equivalent to the problem

$$
u(t)=u_{0}+\sum_{s=t_{0}}^{t-1} g(s, u(s))
$$

By summation convention $\sum_{s=t_{0}}^{t_{0}-1} g(s, u(s))=0$ and so $u(t)$ given above is the solution of (2).
Definition: 2.2 Let $r(t)$ be any solution of (1) on $J$. Then is said to be maximal solution of (1), if every solution of $x(t)$ of (1) existing on $J$, the inequality $x(t) \leq r(t)$ holds for $t \in J$.

Definition: 2.2 A solution $\rho(t)$ of (1) is said to be minimal solution of (1), $\rho(t) \leq x(t)$ holds for $t \in J$. Author proved following theorem in [5].
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Theorem: 2.2[5] Assume that
(i) $K: J \times J \times R \rightarrow R$ and $K(t, s, x)$ is nondecreasing in $x$ for each fixed $(t, s)$ and one of the inequalities

$$
\begin{aligned}
& x(t) \leq h(t)+\sum_{s=t_{0}}^{t-1} K(t, s, x(s)) \\
& y(t) \geq h(t)+\sum_{s=t_{0}}^{t-1} K(t, s, y(s))
\end{aligned}
$$

is strict where $x, y: J \rightarrow R$;
(ii) $x\left(t_{0}\right)<y\left(t_{0}\right)$.

Then

$$
x(t)<y(t), \quad t \geq t_{0} .
$$

## 3. COMPARISON RESULTS:

In this section we obtain the comparison results on solution of (1).
Theorem: 3.1 Let $G: J \times J \times R_{+} \rightarrow R_{+}$is continuous, $G(t, s, u)$ is monotone nondecreeasing in $u$ for each $(t, s)$ and

$$
m(t) \leq m_{0}(t)+\sum_{s=t_{0}}^{t-1} G(t, s, m(s)), \quad t \geq t_{0}
$$

where $m: J \rightarrow R$ is continuous. Suppose that $r(t)$ is the maximal solution of the summation equation

$$
\begin{equation*}
u(t)=u_{0}(t)+\sum_{s=t_{0}}^{t-1} G(t, s, u(s)) \tag{3}
\end{equation*}
$$

existing on J. Then the inequality $m\left(t_{0}\right) \leq \mathrm{u}_{0}\left(t_{0}\right)$ implies

$$
\begin{equation*}
m(t) \leq r(t), \quad t \geq t_{0} \tag{4}
\end{equation*}
$$

Proof: Let $u(t, \epsilon)$ be any solution of summation equation

$$
u(t)=u_{0}(t)+\epsilon+\sum_{s=t_{0}}^{t-1} G(t, s, u(s))
$$

for $\epsilon>0$ sufficiently small. Since

$$
\lim _{\epsilon \rightarrow 0} x(t, \epsilon) \equiv r(t)
$$

it is enough to show that

$$
m(t)<u(t, \epsilon), \quad t \geq t_{0}
$$

Observe that $m\left(t_{0}\right)<u\left(t_{0}, \epsilon\right)$ and $u(t, \in)>u_{0}(t)+\sum_{s=t_{0}}^{t-1} G(t, s, u(t, s, \in))$.
Hence an application of Theorem 2.4 shows that the inequality (5) is valid. This establishes the theorem.
We shall prove an extension of the result of Theorem 3.1 to systems of summation inequalities. The proof of that will be presented using partial ordering in $R^{n}$.

Let us introduce the relation $\leq$ in $R^{n}$, namely, we set, for any two elements $x, y \in R^{n}$,
$x \leq y$ if and only if $x_{i} \leq y_{i}$ for each $i=1,2, \ldots, n$.
This relation induces a partial ordering in $R^{n}$ and it is easy that, for any bounded set $A \subset R^{n}$, there exists the sup $A$ with respect to the relation (6), which is

$$
\begin{equation*}
\sup A=\min \left[z \in R^{n}: x \leq z \text { for each } x \in A\right] . \tag{7}
\end{equation*}
$$

## Dr. K. L. Bondar*/ Comparison Theorems for Summation Equations /RJPA-1(7), Oct.-2011, Page: 167-170

In fact, we need (7) only for two elements sets, in which case we have

$$
\begin{equation*}
\sup [x, y]=z=\left(z_{1}, z_{2}, \ldots, z_{n}\right), \tag{8}
\end{equation*}
$$

where $z_{i}=\max \left(x_{i}, y_{i}\right), x_{i}, y_{i}$ being the components of $x$ and $y$, respectively. We are now in a position to prove the following result.

Theorem: 3.2 Let $K: J \times J \times R^{n} \rightarrow R^{n}$ is continuous, $K(t, s, x)$ is monotonic nondecreasing in $x$ for each $(t, s)$ and

$$
\begin{equation*}
x(t) \leq x_{0}(t)+\sum_{s=t_{0}}^{t-1} K(t, s, x(s)) \tag{9}
\end{equation*}
$$

where $x, x_{0}: J \rightarrow R^{n}$ is continuous. Suppose that $r(t)$ is the maximal solution of the summation equation

$$
\begin{equation*}
u(t)=x_{0}(t)+\sum_{s=t_{0}}^{t-1} K(t, s, u(s)) \tag{10}
\end{equation*}
$$

existing on J. Then

$$
\begin{equation*}
x(t) \leq r(t), \quad t \geq t_{0} \tag{11}
\end{equation*}
$$

Proof: Define

$$
\begin{equation*}
F(t, s, y)=K(t, s, \sup [y, x(t)]) \tag{12}
\end{equation*}
$$

By (8), $x(t) \leq \sup [y, x(t)]$ and therefore it follows, from the monotonicity of $K$ and (12), that

$$
\begin{equation*}
F(t, s, y) \geq K(t, s, \mathrm{x}(t)) \text { for each } y . \tag{13}
\end{equation*}
$$

Let $r^{*}(t)$ be a maximum solution of

$$
u(t)=x_{0}(t)+\sum_{s=t_{0}}^{t-1} F(t, s, u(s))
$$

existing on $J$. Then using (13) and (9), we get

$$
\begin{aligned}
r^{*}(t) & =x_{0}(t)+\sum_{s=t_{0}}^{t-1} F\left(t, s, r^{*}(s)\right) \\
& \geq x_{0}(t)+\sum_{s=t_{0}}^{t-1} K(t, s, x(s))
\end{aligned}
$$

Hence

$$
\begin{equation*}
r^{*}(t) \geq x(t) \tag{14}
\end{equation*}
$$

It then results from (14) and (8) that

$$
\sup \left[r^{*}(t), x(t)\right]=K\left(t, s, r^{*}(t)\right)
$$

and consequently, by (12),

$$
\left.F\left(t, s, r^{*}(t)\right)=K\left(t, s, r^{*}(t)\right]\right)
$$

Thus $r^{*}(t)$ is also the maximal solution of (10). Hence (14) proves the desired result (11). The proof is complete.
Corollary: 3.3 Let $f: J \times R^{n} \rightarrow R^{n}$ is continuous, $(t, x)$ is monotonic nondecreasing in $x$ for each $t$ and

$$
x(t) \leq x_{0}(t)+\sum_{s=t_{0}}^{t-1} f(s, x(s))
$$

where $x: J \rightarrow R^{n}$ is continuous. Suppose that $r(t)$ is the maximal solution of
existing on J. Then

$$
\begin{aligned}
& \Delta y(t)=f(t, y), \quad y\left(t_{0}\right)=x_{0} \\
& x(t) \leq r(t), \quad t \geq t_{0}
\end{aligned}
$$

## Dr. K. L. Bondar*/Comparison Theorems for Summation Equations/RJPA-1(7), Oct.-2011, Page: 167-170

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