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EXISTENCE AND CHARACTERIZATION OF TIME-DEPENDENT GLOBAL ATTRACTORS FOR A CLASS OF NONCLASSICAL PARABOLIC EQUATIONS WITH CRITICAL EXPONENT

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ABSTRACT

This article is concerned with the longtime behavior for the nonclassical parabolic equation

$$\varepsilon_1(t)u_t-\varepsilon_2(t)\Delta u_t-\Delta u+\varphi(u)=g(x), \qquad (*)$$

on a bounded domain $\Omega\subset\mathbb{R}^N(N\geq 3)$. *We first obtain the existence and characterization of time-dependent global attractors* $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$ *in the time-dependent space* Ξ_t *while the nonlinearity* φ *satisfying critical exponent growth, and then, we prove the optimal regularity of the time-dependent global attractor* $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$, *i.e.,* A_t *is bounded in* Ξ_t^1, *with a bound independent of* t.

Key words: *Nonclassical parabolic equations; time-dependent global attractor; time-dependent absorbing set; critical exponent; regularity.*

Classification: *AMS (2000) 35B41; 35Q35.*

1. INTRODUCTION

The study of the global attractor in autonomous problems has been developed extensively over the last few decades and has become a classical theory with nice works, see, for instance, [3, 4-6, 8, 12-15, 17-19, 21] and the references therein. The global attractor is a natural mathematical object that describes the stationary state of the system and all the possible dynamics, giving crucial information on the long time behavior. In 2013, M. Conti, V.Pata and R. Temam exploit a new framework to introduce the time-dependent global attractor in time-dependent spaces in [7], and it is a good mathematical object to study the longtime dynamics of PDEs with time-dependent coefficient, see, e.g., [7, 10, 11] and the references therein.

Let us consider the following time-dependent coefficient nonclassical parabolic equation

$$\begin{cases}\varepsilon_1(t)u_t-\varepsilon_2(t)\Delta u_t-\Delta u+\varphi(u)=g(x), in \quad \Omega\times[\tau,\infty),\tau\in\mathbb{R},\\ u=0, on \quad \partial\Omega,\\ u(x,\tau)=\phi(x), x\in\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$, $\phi:\Omega\to\mathbb{R}$ is assigned data, $\varepsilon_i(t)$ be a positive decreasing bounded function satisfying

$$\lim_{t\to+\infty}\varepsilon_i(t)=0, \varepsilon_i'(t)<0, i=1,2, \tag{1.2}$$

and there exist positive constants ρ_i and ϱ_i such that

$$\rho_i\leq\varepsilon_i(t)\leq\varrho_i, i=1,2. \tag{1.3}$$

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The nonlinearity $\varphi \in \mathbb{C}(\mathbb{R})$ with $\varphi(0)=0$, and satisfies

$$\left|\varphi(u)-\varphi(v)\right| \le \kappa \left|u-v\right|(1+\left|u\right|^{\frac{N+2}{N-2}-1}+\left|v\right|^{\frac{N+2}{N-2}-1}) \text{, for some } \kappa \ge 0, \tag{1.4}$$

along with the dissipation condition

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1, \tag{1.5}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $g(x) \in L^2(\Omega)$.

The classical reaction diffusion equation has strong background in mathematical physics, and it is very natural in many mathematical models. This problem arises in hydrodynamics and the heat transfer theory, such as heat transfer in a solid in contact with a moving fluid, thermoelastic distortion, diffusion phenomena, heat transfer in two media, problems in fluid dynamics etc. e.g. see [1, 2, 6, 12, 18]. In 1980, E. C. Aifantis in [1] pointed out that the classical reaction-diffusion equation does not contain each aspect of the reaction-diffusion problem, and it neglects viscidity, elasticity and pressure of medium in the process of solid diffusion and so on. In the sequel, Aifantis found out that the energy constitutional equation revealing the diffusion process is different along with the different property of the diffusion solid. Therefore, he constructed the mathematic model by some concrete examples, which contains viscidity, elasticity and pressure of medium that is the nonclassical parabolic equation. Here $\varepsilon_1(t)$ is the density of the fluid, and $-\Delta u_t$ denote the pressure, viscoelasticity and memory, e.g. see [1, 2, 24].

In the case when $\varepsilon_1(t)=\varepsilon_2(t)=1$ is a positiveconstant, the asymptotic behavior of solutions to Eq.(1.1) has been extensively studied by several authors in [16, 20, 22-25] and references therein. When $\varepsilon_1(t)$ and $\varepsilon_2(t)$ depending on time, to our best knowledge, the longtime behavior for the nonclassical equation have not been considered by predecessors. In this article, we borrow some ideas from many literatures: F. Flandoli and B. Schmalfuss in [11] who introduced a family of metric spaces depending on a parameter and applied to the stochastic Navier-Stokes equations with multiplicative white noise; T. Caraballo concerned a one-parameter family of Banach spaces in the context of cocycles for non-autonomous and random dynamical systems in [5] and time-dependent spaces [4] in the context of stochastic partial differential equations; M. Conti, V. Pata and R. Temam in [8] introduced the theory of time-dependent global attractors and apply the theory to the wave equations. In this paper, we apply the abstract theory to a new model Equ.(1.1) to prove the existence of time-dependent global attractors.

Since Equation (1.1) contains the term $-\Delta u_t$, it is different from the usual reaction-diffusion equation essentially. For example, the reaction diffusion equation has some smoothing effect, e.g., although the initial data only belongs to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, for Equation (1.1), if the initial data u_τ belongs to Ξ_t, then the solution $u(t,x)$ with u_τ is always in Ξ_t and has no higher regularity because of $-\Delta u_t$. In this paper, we apply the techniques introduced in [8] to overcome some difficulties, and then establish the asymptotic regularity of solutions.

This paper is organized as following: In Section 1, we have expounded on research progress as regards our research problem, and given some assumptions. In Section 2, we introduce some notations and functions spaces, and we give some useful lemmas. In Section 3, we prove the existence and characterization of time-dependent global attractor for the nonclassical parabolic equation, and the regularity of time-dependent global attractor is stated and proved in Section 4.

Our main results are Theorem 3.1 and Theorem 4.1.

2. PRELIMINARIES

The following notation will be used throughout this paper: $H = L^2(\Omega)$, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For $0 \le \sigma \le 2$, we define the hierarchy of (compactly) nested Hilbert spaces

$$H_\sigma = D(A^{\frac{\sigma}{2}}), \langle \omega, v\rangle_\sigma = \langle A^{\frac{\sigma}{2}}\omega, A^{\frac{\sigma}{2}}v\rangle, \|\omega\|_\sigma = \left\|A^{\frac{\sigma}{2}}\omega\right\|.$$

Then, for $t \in \mathbb{R}$ and $0 \le \sigma \le 2$, we introduce the time-dependent spaces $\Xi_t^\sigma = H_{\sigma+1}$ endowed with the time-dependent product norms

$$\|u\|_{\Xi_t^\sigma}^2 = \varepsilon_1(t)\|u\|_\sigma^2 + \varepsilon_2(t)\|u\|_{\sigma+1}^2.$$

The symbol sigma \$ is always omitted whenever zero, In particular, the time-depended phase space where we settle the problem is

$$\Xi_t = H_1 \text{ with } \|u\|_{\Xi_t}^2 = \varepsilon_1(t)\|u\|^2 + \varepsilon_2(t)\|u\|_1^2$$

Then, we have the compact embeddings

$$\Xi_t^\sigma \subset \Xi_t, 0 \le \sigma \le 2$$

with injection constants independent of $t \in \mathbb{R}$.

Note that the spaces Ξ_t are all the same as linear spaces, and the norm $\|z\|_{\Xi_t}^2$ and $\|\cdot\|_{\Xi_t}^2$ are equivalent for any fixed t, $\tau \in \mathbb{R}$ .Now, we iterate some definitions and abstract results concerning the time-dependent global attractor, which is necessary to obtain our main results, please refer the reader to see [7, 9]for more details.

Definition 2.1: A time-dependent absorbing set for the process $U(t,\tau)$ is a uniformly bounded family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R \ge 0$ there exists $\theta_e = \theta_e(R) \ge 0$ such that

$$\tau \le t - \theta_e \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset B_t.$$

For $t \in \mathbb{R}$, let X_t be a family of normed spaces, we consider the collection

$$\mathbb{K} = \{\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}}\} : K_t \subset X_t \ compact, \mathfrak{K} \ pullback \ attracting\}.$$

When $\mathbb{K} \neq \phi$ we say that the process is asymptotically compact.

Definition 2.2: We call a time-dependent global attractor the smallest element of $\mathbb{K}$, i.e. the family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}} \in \mathbb{K}$ such that $A_t \subset K_t, \forall t \in \mathbb{R}$,for any element $\mathfrak{K} = \{K_t\}_{t\in\mathbb{R}} \in \mathbb{K}$.

Definition 2.3: If $U(t,\tau)$ is asymptotically compact, then the time-dependent attractor $\mathfrak{A}$ exists and coincides with the set $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$. In particular, it is unique.

Definition 2.4: If $U(t,\tau)$ is a T -closed process for some $T > 0$, which possesses a time-dependent global attractor $\mathfrak{A}$, then $\mathfrak{A}$ is invariant.

Remark 2.1: If the process $U(t,\tau)$ is closed it is T -closed, for any $T > 0$. Note that if the process $U(t,\tau)$ is a continuous (or even norm-to-weak continuous) map for all $t \ge \tau$, then the process is closed.

Definition 2.5: A function $z : t \mapsto z(t) \in \Xi_t$ is a complete bounded trajectory (CBT) of $U(t,\tau)$ if and only if

$$\sup_{t\in\mathbb{R}} \|z(t)\| \Xi_t < \infty$$

and

$$z(t) = U(t,\tau)z_\tau, t \ge \tau, \tau \in \mathbb{R}.$$

Definition 2.6: Let $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ be then the time-dependent attractor of $U(t,\tau)$. If $\mathfrak{A}$ is invariant, then

$$A_t = \{z(t) = \Xi_t : z \ CBT \ of \ U(t,\tau)\}.$$

Accordingly, we can write

$$\mathfrak{A} = \{z : t \mapsto z(t) \in \Xi_t \ with \ z \ CBT \ of \ U(t,\tau)\}.$$

3. EXISTENCE AND CHARACTERIZATION OF THE TIME-DEPENDENT GLOBAL ATTRACTOR

3.1 Well-posedness

For any $\tau \in \mathbb{R}$, we rewrite problem (1.1) as

$$\begin{cases} \varepsilon_1(t)u_t + \varepsilon_2(t)Au_t + Au + \varphi(u) = g(x), \ t > \tau, \\ u(x,\tau) = \phi \end{cases} \quad (3.1)$$

Applying the Galerkin approximation scheme, we can obtain the following result concerning the existence and uniqueness of solutions easily, see e.g. [14, 16, 23, 24].

Lemma 3.1: Under the assumptions of (1.2)-(1.5), for any $\phi \in \Xi_\tau$, there is a unique solution u of (3.1) satisfy, on any interval $[\tau, t]$ with $t \geq \tau$,

$$u \in \mathcal{C}([\tau, t]; \Xi_1).$$

Furthermore, for $i = 1, 2$, let $u_\tau^i \in \Xi_\tau$ be two initial conditions such that $\|u_\tau^i\|_{\Xi_\tau} \leq R$, and denote by u_i the corresponding solutions to problem (3.1). Then estimates hold as follows:

$$\|u_1(t) - u_2(t)\|_{\Xi_t}^2 \leq e^{\delta(t-\tau)} \|u_\tau^1 - u_\tau^2\|_{\Xi_t}^2, \quad t \geq \tau, \tag{3.2}$$

for some constant $\delta = \delta(R) \geq 0$.

Based on Lemma 3.1 above, we can define a family of maps with $t \geq \tau \in \mathbb{R}$

$$U(t,\tau): \Xi_\tau \to \Xi_t,$$

acting as

$$u_\tau \to u(t) = U(t,\tau)u_\tau$$

where u is the unique solution to (3.1) with initial time τ and initial condition $u_\tau \in \Xi_\tau$, defines a strongly continuous process on the family $\{\Xi_t\}_{t\in\mathbb{R}}$.

3.2 Time-dependent absorbing set

Lemma 3.2: Under the assumptions of (1.2)-(1.5). For $u_\tau \in \Xi_\tau$, $t \geq \tau$, let $U(t,\tau)u_\tau$ be the solution of (3.1), then, there exist positive constants δ, $C_{\delta,\nu,\|g\|}$ and an increasing positive function $\mathcal{Q}$ such that

$$\|U(t,\tau)u_\tau\|_{\Xi_t} \leq \mathcal{Q}(\|u_\tau\|_{\Xi_\tau})e^{-\delta(\tau-\tau)} + C_{\delta,\nu,\|g\|}, \quad \forall t \geq \tau. \tag{3.3}$$

Proof: Multiplying (3.1) by u, we obtain

$$\frac{1}{2}\frac{d}{dt}(\varepsilon_1(t)\|u\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}u\|^2) + \|A^{\frac{1}{2}}u\|^2 + \langle\varphi(u),u\rangle = \langle g(u),u\rangle + \frac{\varepsilon_1'}{2}\|u\|^2 + \frac{\varepsilon_2'}{2}\|A^{\frac{1}{2}}u\|^2. \tag{3.4}$$

According to (1.5), the following inequalities hold for some $0 < \nu < 1$,

$$\langle\varphi(u),u\rangle \geq -(1-\nu)\|A^{\frac{1}{2}}u\|^2 - C. \tag{3.5}$$

Using (2.3) and Young and Poincaré inequalities, for some positive constants $\delta = \delta_{\nu,\rho_1,\rho_2,\lambda_1}$ small enough, and noting that $\varepsilon_i' < 0 (i = 1, 2)$, we get

$$\frac{d}{dt}(\varepsilon_1(t)\|u\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}u\|^2) + \delta(\varepsilon_1(t)\|u\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}u\|^2) < C_{\delta,\rho_1,\rho_2,\|g\|}. \tag{3.6}$$

By the Gronwall lemma, we infer

$$\|u(t)\|_{\Xi_t}^2 \leq Ce^{-\delta(t-\tau)}\|u(\tau)\|_{\Xi_t}^2 + C_{\delta,\rho_1,\rho_2,\|g\|}. \tag{3.7}$$

This completes the proof.

Lemma 3.3: (Time-dependent absorbing set) Under the assumptions of (1.2)-(1.5), there exists a constant $R_1 > 0$, such that the family $\mathfrak{B} = \{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for $U(t,\tau)$.

Indeed, according to the proof of Lemma 3, 2, for $u \in \mathbb{B}_\tau(R)$, there exists $\theta_e \geq 0$, provided that $t - \tau \geq \theta_e$,

$$\|U(t,\tau)u_\tau\|_{\Xi_t} \leq R_0. \tag{3.8}$$

This concludes the existence of the time-dependent absorbing set.

We can assume that the time-dependent absorbing set $\mathfrak{B}_t = \mathbb{B}_t(R_1)$ is positively invariant (namely $U(t,\tau)B_\tau \subset B_t$ for all $t \geq \tau$). In fact, calling θ_e the entering time of B_t such that

$$U(t,\tau)B_\tau \subset B_t, \ \ \forall \tau \le t-\theta_e .$$

we can substitute B_t with the invariant absorbing family

$$\bigcup_{\tau \le t-\theta_e} U(t,\tau)B_\tau \subset B_t .$$

3.3. Time-dependent global attractor

Based on Definition 2.2, the existence of the time-dependent global attractor can be obtained by a direct application of the abstract Theorem 2.3. Precisely, in order to show that the process is asymptotically compact, we shall exhibit a pullback attracting family of compact sets. To this aim, the strategy classically consists in finding a suitable decomposition of the process in the sum of a decaying part and of a compact one.

3.4. The first decomposition of the system equations

For the nonlinearity φ, following [7, 8], we can decompose φ as follows:

$$\varphi = \varphi_0 + \varphi_1 ,$$

where $\varphi_0, \varphi_1 \in \mathcal{C}(\mathbb{R})$ satisfy, for some $c \ge 0$,

$$|\varphi_0(u)-\varphi_0(v)| \le \kappa |u-v| \left(1+|u|^{\frac{N+2}{N-2}-1} + |v|^{\frac{N+2}{N-2}-1} \right), \quad \forall u,v \in \mathbb{R} , \tag{3.9}$$

$$\varphi_0(u)u \ge 0, \quad \forall u \in \mathbb{R} , \tag{3.10}$$

$$\liminf_{|u|\to\infty} \frac{\varphi_1(u)}{u} > -\lambda_1 , \tag{3.11}$$

$$|\varphi_1(u)| \le c(1+|u|), \quad \forall u \in \mathbb{R} , \tag{3.12}$$

Noting that $\mathfrak{B} = \{\mathbb{B}_t(R_1)\}_{t\in\mathbb{R}}$ is a time-dependent absorbing set for $U(t,\tau)u_\tau$, then for each initial data $u_\tau \in \mathbb{B}_t(R_1)$, we decompose $U(t,\tau)$ as

$$U(t,\tau)u_\tau = T(t,\tau)u_\tau + S_g(t,\tau)u_\tau .$$

where $v = T(t,\tau)u_\tau$ and $\omega = S_g(t,\tau)u_\tau$ solves the following equations respectively:

$$\begin{cases} \varepsilon_1(t)v_t + \varepsilon_2(t)Av_t + Av + \varphi_0(v) = 0, \\ T(\tau,\tau) = u_\tau , \end{cases} \tag{3.13}$$

and

$$\begin{cases} \varepsilon_1(t)\omega_t + \varepsilon_2(t)A\omega_t + A\omega + \varphi(u) - \varphi_0(v) = g, \\ S_g(\tau,\tau) = 0, \end{cases} \tag{3.14}$$

Lemma 3.4: Under the assumptions of (1.2)-(1.5), (3.9)-(3.12), there exists $\delta = \delta(\mathfrak{B}) > 0$ such that

$$\|T(t.\tau)u_\tau\|_{\Xi_t} \le C_3 e^{-\delta(t-\tau)}, \quad \forall t \ge \tau . \tag{3.15}$$

Proof: Multiplying (3.13) by υ and integrating over Ω, we infer

$$\frac{1}{2}\frac{d}{dt}(\varepsilon_1(t)\|v\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}v\|^2) + \|A^{\frac{1}{2}}v\|^2 + \langle \varphi_0(v), v\rangle = \frac{\varepsilon_1'}{2}\|v\|^2 + \frac{\varepsilon_2'}{2}\|A^{\frac{1}{2}}v\|^2 . \tag{3.16}$$

Note that $\varepsilon_i' < 0 (i=1,2)$, and using (1.3) and (3.10), by Young and Poincaré inequalities, for $\delta > 0$ small, we infer

$$\frac{d}{dt}(\varepsilon_1(t)\|v\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}v\|^2) + \delta(\varepsilon_1(t)\|v\|^2 + \varepsilon_2(t)\|A^{\frac{1}{2}}v\|^2) \le 0. \tag{3.17}$$

By the Gronwall lemma, we complete the proof.

Remark 3.5: From Lemma 3.2 and Lemma 3.4, we have the uniform bound

$$\sup_{t\ge\tau}[\|U(t,\tau)\|_{\Xi_t} + \|T(t,\tau)\|_{\Xi_t} + \|S_g(t,\tau)\|_{\Xi_t}] \le C.$$

Lemma 3.6: Under the assumptions of (1.2)-(1.5), (3.9)-(3.12), for every $T > 0$, there exists $M = M(\mathfrak{B}) > 0$ such that

$$\sup_{t\ge\tau}\|S_g(T,\tau)u_\tau\|_{\Xi_t^{\frac{1}{3}}} \le M. \tag{3.18}$$

Proof: Multiplying (3.14) by $A^{\frac{1}{3}}\omega$ and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\varepsilon_1(t)\|A^{\frac{1}{6}}\omega\|^2 + \varepsilon_2(t)\|A^{\frac{2}{3}}\omega\|^2) + \|A^{\frac{2}{3}}\omega\|^2 - \frac{\varepsilon_1'}{2}\|A^{\frac{1}{6}}\omega\|^2 - \frac{\varepsilon_2'}{2}\|A^{\frac{2}{3}}\omega\|^2\\ &= -\langle\varphi(u) - \varphi_0(v) - \mathfrak{g}, A^{\frac{1}{3}}\omega\rangle\\ &= -\langle\varphi(u)-\varphi(v), A^{\frac{1}{3}}\omega\rangle - \langle\varphi_1(v), A^{\frac{1}{3}}\omega\rangle + \langle\mathfrak{g}, A^{\frac{1}{3}}\omega\rangle.\end{aligned} \tag{3.19}$$

By Remark 3.1 and (3.9)-(3.12), and noting that $\frac{2}{N} + \frac{3N-8}{6N} + \frac{3N-4}{6N} = 1$, we infer

$$\begin{aligned}\left|\langle\varphi(u)-\varphi(v), A^{\frac{1}{3}}\omega\rangle\right| &\le c\int_\Omega (1+|u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}})|\omega||A^{\frac{1}{3}}\omega|dx\\ &\le c(1+\|u\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}} + \|v\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}})\|\omega\|_{L^{\frac{6N}{3N-8}}}\|A^{\frac{1}{3}}\omega\|_{L^{\frac{6N}{3N-4}}}\\ &\le c(1+\|A^{\frac{1}{2}}u\|^{\frac{4}{N-2}} + \|A^{\frac{1}{2}}v\|^{\frac{4}{N-2}})\|A^{\frac{2}{3}}\omega\|^2\\ &\le c\|A^{\frac{2}{3}}\omega\|^2,\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\left|\langle\varphi_1(v), A^{\frac{1}{3}}\omega\rangle\right| &\le c\int_\Omega (1+|v|)|A^{\frac{1}{3}}\omega|dx\\ &\le C\|A^{\frac{2}{3}}\omega\|,\end{aligned} \tag{3.21}$$

And

$$\langle g, A^{\frac{1}{3}}\omega\rangle \le c\|g\|\|A^{\frac{1}{3}}\omega\|, \tag{3.22}$$

where we have used the continuous embedding $H_1 = D(A^{\frac{1}{2}}) \mapsto L^{\frac{2N}{N-2}}(\Omega)$,

$$H_{\frac{2}{3}} = D(A^{\frac{1}{3}}) \mapsto L^{\frac{6N}{3N-4}}(\Omega), H_{\frac{4}{3}} = D(A^{\frac{2}{3}}) \mapsto L^{\frac{6N}{3N-8}}(\Omega).$$

By Young and Poincaré inequalities, for $\delta > 0$ small, we infer

$$\frac{d}{dt}(\varepsilon_1(t)\|A^{\frac{1}{6}}\omega\|^2 + \varepsilon_2(t)\|A^{\frac{2}{3}}\omega\|^2) \le C(\varepsilon_1(t)\|A^{\frac{1}{6}}\omega\|^2 + \varepsilon_2(t)\|A^{\frac{2}{3}}\omega\|^2). \tag{3.23}$$

By use of the Gronwall lemma and noting that $S_g(\tau,\tau) = 0$, we complete the proof.

Theorem 3.1: (Existence and characterization of the time-dependent global attractor) Under the assumptions of (1.2)-(1.5), the process $U(t,\tau):\Xi_\tau \to \Xi_t$ generated by problem (1.1) admits an invariant time-dependent global attractor $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$.

Furthermore, we can write

$$\mathfrak{A}=\{z:t\mapsto z(t)\in\Xi_t \ \ with\ z\ CBT\ of\ U(t,\tau)\}.$$

Proof: According to Lemma 3.6, we consider the family $\mathfrak{K}=\{K_t\}_{t\in\mathbb{R}}$ where

$$K_t=\{u\in\Xi_t^{\frac{1}{3}}:\|u\|_{\Xi_t^{\frac{1}{3}}}\le M\}$$

K_t is compact by the compact embedding $\Xi_t^{\frac{1}{3}}\in\Xi_t$; besides, since the injection constants are independent of t, $\mathfrak{K}$ is uniformly bounded. Hence, according to Lemma 3.3, Lemma 3.4 and Lemma 3.6, $\mathfrak{K}$ is pullback attracting, and the process $U(t,\tau)$ is asymptotically compact, which proves the existence of the unique time-dependent global attractor. In order to state the invariance of the time-dependent global attractor, due to the strong continuity of the process $U(t,\tau)$ stated in Lemma 3.1, according to Remark 2.1, the process $U(t,\tau)$ is closed, and it is T-closed, for some $T>0$, then by Theorem 2.4, we know that the time-dependent global attractor $\mathfrak{A}$ is invariant. By use of Theorem 2.6, we know that the time-dependent global attractor can be characterized as sections of the set of complete bounded trajectories.

This completes the proof.

From Lemma 3.6 and Theorem 3.1, we immediately have the following regularity result:

Remark 3.7: $\mathcal{A}_t$ is bounded in $\Xi_t^{\frac{1}{3}}$ (with a bound independent of t).

4. REGULARITY OF THE TIME-DEPENDENT GLOBAL ATTRACTOR

II. The second decomposition of the system equations

We fix $\tau\in\mathbb{R}$, and each initial data $u_\tau\in\mathcal{A}_\tau$, decomposing $U(t,\tau)$ as

$$U(t,\tau)u_\tau=V(t,\tau)u_\tau+W_g(t,\tau)u_\tau,$$

where $v=V(t,\tau)u_\tau$ and $\omega=W(t,\tau)u_\tau$ solves the following equations respectively:

$$\begin{cases}\varepsilon_1(t)v_t+\varepsilon_2(t)Av_t+Av=0,\\ V(\tau,\tau)=u_\tau,\end{cases} \tag{4.1}$$

and

$$\begin{cases}\varepsilon_1(t)\omega_t+\varepsilon_2(t)A\omega_t+A\omega+\varphi(u)=g,\\ W_g(\tau,\tau)=0,\end{cases} \tag{4.2}$$

As a particular case of Lemma 3.4, we learn that

$$\|V(t,\tau)u_\tau\|_{\Xi_t}\le C_4e^{-\delta(t-\tau)},\ \ \forall t\ge\tau. \tag{4.3}$$

Lemma 4.1: Under the assumptions of (1.2)-(1.5), for some $M=M(\mathfrak{A})>0$, We have the uniform bound

$$\sup_{t\ge\tau}\|W_g(t,\tau)u_\tau\|_{\Xi_t^1}\le M. \tag{4.4}$$

Proof: We denote

$$\xi(t)=\varepsilon_1(t)\|A^{\frac{1}{2}}\omega\|^2+\varepsilon_2(t)\|A\omega\|^2+\|A\omega\|^2-2\langle g,A\omega\rangle,$$

by (1.3), we have

$$c\|\omega\|_{\Xi_t^1}^2-C_6\le\xi(t)\le c\|\omega\|_{\Xi_t^1}^2+C_7. \tag{4.5}$$

Multiplying (4.2) by $A\omega_t + A\omega$, we obtain

$$\begin{aligned}&\frac{d}{dt}\xi(t)+2\varepsilon_1(t)\left\|A^{\frac{1}{2}}\omega_t\right\|^2+2\varepsilon_2(t)\left\|A\omega_t\right\|^2+2\left\|A\omega\right\|^2+2\langle\varphi(u),A\omega\rangle-2\langle g,A\omega\rangle\\&+2\langle\varphi(u),A\omega_t\rangle=\varepsilon_1'\left\|A^{\frac{1}{2}}\omega\right\|^2+\varepsilon_2'\left\|A\omega\right\|^2.\end{aligned} \tag{4.6}$$

Noting that (1.2) and (1.3), and using Young and Poincaré inequalities, for $\delta=\delta_{\rho_1,\rho_2,\lambda_1}>0$ small, we infer

$$\begin{aligned}&\frac{d}{dt}\xi(t)+\delta\xi(t)+2\varepsilon_1(t)\left\|A^{\frac{1}{2}}\omega_t\right\|^2+2\varepsilon_2(t)\left\|A\omega_t\right\|^2\\&\qquad\le-2\langle\varphi(u),A\omega\rangle-(2+2\delta)\langle g,A\omega\rangle-2\langle\varphi(u),A\omega_t\rangle.\end{aligned} \tag{4.7}$$

Denoting by $C>0$ a generic constant depending on the size of A_t in $\Xi_t^{\frac{1}{3}}$, using the invariance of the attractor, we find

$$\left\|U(t,\tau)u\right\|_{\Xi_t^{\frac{1}{3}}}\le C.$$

Exploiting the embeddings $\Xi_t^{\frac{1}{5}}=H_{\frac{6}{5}}=D(A^{\frac{3}{5}})\subset L^{\frac{10N}{5N-12}}(\Omega)$ and noting that $\frac{5N-12}{10N}\cdot\frac{N+2}{N-2}<\frac{1}{2}$, we get

$$\begin{aligned}-2\langle\varphi(u),A\omega\rangle&\le C\int_\Omega(1+|u|^{\frac{N+2}{N-2}})|A\omega|\\&\le C(1+\|u\|_{L^{\frac{10N}{5N-12}}}^{\frac{N+2}{N-2}})\|A\omega\|\\&\le\frac{\delta}{4}\|A\omega\|^2+C,\end{aligned} \tag{4.8}$$

and

$$-(2+2\delta)\langle g,A\omega\rangle\le\frac{\delta}{2}\|A\omega\|^2+C_{\delta,\|g\|}. \tag{4.9}$$

Similar to (4.8), we have

$$-2\langle\varphi(u),A\omega_t\rangle\le2\|A\omega_t\|^2+C. \tag{4.10}$$

It follows from (4.6)-(4.10) that

$$\frac{d}{dt}\xi(t)+\frac{\delta}{2}\xi(t)\le C. \tag{4.11}$$

By the Gronwall lemma, and noting that(4.5), we can get (4.4) immediately.

This completes the proof.

Therefore, we have the following regularity result.

Theorem 4.1: (Regularity of the time-dependent global attractor) Under the assumptions of (1.2)-(1.5), the time-dependent global attractor $\mathfrak{A}=\{A_t\}_{t\in\mathbb{R}}$, A_t is bounded in Ξ_t^1, with a bound independent of t.
Indeed, we denote

$$\Gamma_t=\{u\in\Xi_t^1:\|u\|_{\Xi_t^1}\le M_1\}. \tag{4.12}$$

According to inequalities (4.3) and (4.4), for all $t\in\mathbb{R}$, we have

$$\lim_{\tau\to-\infty}dist(U(t,\tau)A_\tau,\Gamma_t)=0. \tag{4.13}$$

where $dist$ denote the Hausdorff semi-distance in Ξ_t, i.e.

$$dist(B_1,B_2)=\sup_{x\in B_1}dist_{\Xi_t}(x,B_2)=\sup_{x\in B_1}\inf_{y\in B_2}\|x-y\|_{\Xi_t}. \tag{4.14}$$

From Theorem 3.1, we know that the time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is invariant, this means

$$dist(A_t, \Gamma_t) = 0. \tag{4.15}$$

Hence, $A_t \subset \overline{\Gamma_t} = \Gamma_t$, i.e., A_t is bounded in Ξ_t^1, with a bound independent of t.

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