International Research Journal of Pure Algebra-7(3), 2017, 509-512
Available online through www.rjpa.info ISSN 2248–9037

Strongly n-Ding projective and injective Modules under Change of Rings

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(Received On: 03-03-17; Revised & Accepted On: 31-01-17)

ABSTRACT

In this paper, we mainly investigate some properties of strongly n-Ding projective and injective modules under the extension of rings, which mainly including excellent extensions, polynomial extension and localizations.

Key words: *excellent extensions; strongly n-Ding projective modules; strongly n-Ding injective modules.*

1. INTRODUCTION

Throughout the paper R is a commutative ring with identity element, and all R - module are unital. If M is any R -module, we use $pd_R(M)$ and $id_R(M)$ to denote projective and injective dimensions of M .

In $[5]$ and $[7]$, the author introduced strongly Gorenste in flat and Gorenste in FP -injective module, which are defined as follows:

Definition: Let n be a positive integer.

(1) An R -module M is called stongly Gorenste in flat module (we called Ding projective module) if there is an exact sequence

$$P: \qquad \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

Of projective right R -module with $M \cong Ker\left(P^0 \to P^1\right)$ such that $Hom_R(-,Q)$ leaves the sequences exact, where Q is a flat R-module.

(2) An R -module M is called Gorenste in FP -injective module (we called Ding injective module) if there is an exact sequence

$$E: \qquad \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

Of injective right R -module with $M \cong Ker\left(E^0 \to E^1\right)$ such that $Hom_R(I,-)$ leaves the sequences exact, where I is an FP -injective module.

The main purpose of this paper is to study some properties of strongly n-Ding projective and injective modules under excellent extensions, polynomial extensions and localizations, respectively and we get some interesting results.

2. STRONGLY n-DINGPROJECTIVE AND INJECTIVE MODULES UNDER CHANGE OF RINGS

We begin with the following lemmas.

Lemma 2.1([11]): An R -module M is called stongly Ding projective module if and only if there exists a short exact sequence $0 \to M \to P \to M \to 0$, where P is projective module and $Ext_R^1(M,F)=0$ for all flat modules F .

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Lemma 2.2([12]): An R-module Miscalled stongly Ding injective module if and only if there exists as hort exact sequence $0 \to M \to E \to M \to 0$, where E is injective module and $Ext_R^1(Q, M) = 0$ for all FP -injective modules Q.

Definition 2.3: An R -module M is called strongly n-Ding projective module (S-n D- projective module for short) if there exists a short exact sequence $0 \to M \to P \to M \to 0$ with $pd_R(P) \le n$ and $Ext_R^{n+1}(M, F) = 0$ for all flat modules F.

Definition 2.4: An R -module M is called strongly n-Ding injective module (S-nD- injective module for short) if there exists a short exact sequence $0 \to M \to E \to M \to 0$ with $id_R(E) \le n$ and $Ext_R^{n+1}(Q, M) = 0$ for all FP - injective modules Q.

Proposition 2.5: Let R be a commutative ring and Q a projective R -module. If M is an S-nD-projective R -module, then $M \otimes Q$ is an S-nD-projective R -module.

Proof: Since M Is an S-nD-projective R-module, there is an short exact sequence $0 \to M \to P \to M \to 0$ with $pd_R(P) \le n$ .Then $0 \to M \otimes Q \to P \otimes Q \to M \otimes Q \to 0$ and $pd_R(P \otimes Q) \le n$ by [9, ch.2, Theorem3]. Let Q' be any flat R -module. Then $Ext_R^i(M \otimes_R Q, Q') = Hom(Q, Ext_R^i(M, Q')) = 0$ by[9,p.256,Lemma9.20] for all $i > n$. Hence $M \otimes Q$ is an S-nD-projective R-module.

Proposition 2.6: Let R and S be equivalent rings via equivalence $F : R$ -Mod→ S -Mod and $G : S$ -Mod→ R -Mod. Then

(1) M is S-nD-projective R -Mod if and only if $F(M)$ is S-nD-projective S -Mod.

(2) M is S-nD-injective R -Mod if and only if $F(M)$ is S-nD-injective S -Mod.

Proof:

(1) Since M is an S-nD-projective R -module, there is the short exact sequence $P : 0 \to M \to P \to M \to 0$, where $pd_R(P) \le n$ .Then $F(P) : 0 \to F(M) \to F(P) \to F(M) \to 0$ is a exact sequence of S -Mod with $pd_R(F(P)) \le n$. Let Q be any flat S -Mod. Then $Ext_S^i(F(M), Q) \cong Ext_R^i(M, G(Q)) = 0$ for all $i > n$. Hence $F(M)$ is S-nD-projective S -Mod. By $GF(M) \cong M$.

(2) By analogy with the proof of (1).

A ring S is said to be an almost excellent extension of a ring R, if the following conditions are satisfied: (1) The ring S is called right R -projective in case for any right S -module M_S with an S -sub module $N_R | M_R$ implies $N_S | M_S$ .For example, every $n \times n$ matrix ring $M_n(R)$ is right R -projective. (2) The ring extension $S \ge R$ is called a finite normalization extension in case there is a finite subset $\{s_1, \cdots, s_n\}$ of S such that $S = \sum_i^n s_i R$ and $s_i R = R s_i$ for $i = 1, \cdots, n$ (3) A finite normalization extension $S \ge R$ is called an excellent extension in case condition (1) is satisfied and ${}_R S$, S_R are free module with a common as is $\{s_1, \cdots, s_n\}$. For example, every $n \times n$ matrix ring $M_n(R)$ is an excellent extension.

Proposition 2.7: Assume that $S \ge R$ is an excellent extension, if ${}_R M$ is a S-nD projective R -mod, then $S \otimes M$ is a S-n D projective S -mod.

Proof: There exists an exact sequence $0 \to M \to P \to M \to 0$ in R -mod with $pd_R(P) \le n$ .Then $0 \to S \otimes M \to S \otimes P \to S \otimes M \to 0$ is exact in S -mod, with $pd_R(S \otimes P) \le n$. Let Q be any flat S -mod, then Q is a flat R -mod, and so $Ext_S^i(S \otimes M, Q) \cong Ext_R^i(M, Q) = 0$ by [9, p.258, 9.21] for all $i > n$. It follows that $S \otimes M$ is a S-nD projective S -mod.

Proposition 2.8: Let R be a ring and let $n \geq 1$ be a natural number. For any $M \in M_n(R)$-Mod. Then

(1) (1) ${}_R M$ is S-nD projective R -mod if and only if $M_n(R) \otimes M$ is S-nD projective R -mod.

(2) (2) ${}_R M$ is S-nD injective R -mod if and only if $Hom_R(M_n(R), M)$ is S-nD injective R -mod.

If R is a ring, then $R[x]$ is the polynomial ring. If M is a left R -module, write $M[x] = R[x] \otimes_R M$ .Since $R[x]$ is a free R -module and since tensor product commutes with sums, we may regard the elements of $M[x]$ a s' vectors' $x^i \otimes_R m_i$, $i \geq 0$. $m_i \in M$ with almost all $m_i = 0$.

Proposition 2.9: Let R be a commutative ring. If M is a S-nD projective R -mod, then $M[x]$ is a S-nD projective $R[x]$ -mod.

Proof: There is an exact sequence $0 \to M \to P \to M \to 0$ in R -mod with $pd_R(P) \leq n$ .so $0 \to M[x] \to P[x] \to M[x] \to 0$ is exact in $R[x]$ -mod and $pd_{R[x]}(P[x]) \leq n$. Let Q be any flat $R[x]$ -mod. Then $Q[x] \cong R[x] \otimes Q \cong R^N \otimes_R Q \cong Q^N$ .Hence $Q[x]$ isa flat $R[x]$ -module, and so Q is a flat R -module. Thus $Ext_{R[x]}^i(M[x], Q) = Ext_R^i(M, Q) = 0$ by [9, p.258, 9.21], for all $i > n$, and hence $M[x]$ is an S-nD projective $R[x]$ -module.

Let R be a commutative ring and S a multiplicatively closed set of R .Then $S^{-1}R = R \times S/\sim = \{a/s \mid a \in R, s \in S\}$ is a ring and $S^{-1}M = M \times S/\sim = \{x/s \mid x \in M, s \in S\}$ is an $S^{-1}R$ -module. If P is a prime ideal of R and $S = R - P$, then we will denote $S^{-1}M$, $S^{-1}R$ by M_P , R_P respectively.

Proposition 2.10: Let R be a commutative ring and S a multiplicatively closed set of R .If M is a S-nD projective R -module, then $S^{-1}M$ is an S-nD projective $S^{-1}R$ -mod.

Proof: Since M is a S-nD projective R -module, there exists an exact sequence $0 \to M \to P \to M \to 0$ in R -mod with $pd_R(P) \leq n$ .Then $0 \to S^{-1}M \to S^{-1}P \to S^{-1}M \to 0$ is exact in $S^{-1}R$ -mod and $pd_{S^{-1}R}(S^{-1}P) \leq n$ .Let $\overline{Q}$ be any flat $S^{-1}R$ -mod, then $\overline{Q}$ is a flat R -module by [11, Lemma 3.20], so

$$Ext_{S^{-1}R}^i(S^{-1}M, \overline{Q}) \cong Ext_{S^{-1}R}^i(S^{-1}R \otimes M, \overline{Q}) \cong Ext_R^i(M, \overline{Q}) = 0 \quad by \ [9, p.258, 9.21] \ for \ all \ \ i > n,$$

Hnece $S^{-1}M$ is an S-nD projective $S^{-1}R$ -mod.

Proposition 2.11: Let R be a commutative ring and S a multiplicatively closed set of R .If $S^{-1}R$ is a projective R -module, then

(1) If M is an S-nD injective R -module, then $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.

(2) For any R -module M , $Hom(S^{-1}R, M)$ is an S-n D injective R -module if and only if $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.

Proof: There is an exact sequence $0 \to M \to E \to M \to 0$ in R -mod with $id_R(E) \leq n$ .Then $0 \to Hom(S^{-1}R, M) \to Hom(S^{-1}R, E) \to Hom(S^{-1}R, M) \to 0$ is exact in $S^{-1}R$ -mod and $id_{S^{-1}R}(Hom(S^{-1}R, E)) \leq n$ by [6, Theorem 3.2.9]. Let $\overline{I}$ be any injective $S^{-1}R$ -module. Then $\overline{I}$ is an injective R-module by [4, lemma1.2].So $Ext_{S^{-1}R}^i(\overline{I}, Hom(S^{-1}R, M)) \cong Ext_R^i(\overline{I}, M) = 0$ by [9, p.258, 9.21] for all $i > n$, and hence $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.

(2) is obvious.

Since $Hom\left(S^{-1}R, M\right)$ is an S-n D injective $S^{-1}R$ -module, then there exists an exact sequence $0 \to Hom\left(S^{-1}R, M\right) \to \overline{E} \to Hom\left(S^{-1}R, M\right) \to 0$ in $S^{-1}R$ - mod with $id_{S^{-1}R}\left(\overline{E}\right) \leq n$ .Then $\overline{E}$ is an injective R -module. Let I be any injective R -module. Then $S^{-1}I$ is an injective $S^{-1}R$ -module. So $Ext_R^i\left(I, Hom_{S^{-1}R}\left(S^{-1}R, Hom_R\left(S^{-1}R, M\right)\right)\right) \cong Ext_{S^{-1}R}^i\left(S^{-1}I, Hom\left(S^{-1}R, M\right)\right) = 0$ by [8, proposition 5.17] and by [9, p.258,9.21] for all $i > n$,and hence $Hom\left(S^{-1}R, M\right)$ is an S-n D injective R -module.

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Source of Support: Nil, Conflict of interest: None Declared