



Strongly n -Ding projective and injective Modules under Change of Rings

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ABSTRACT

In this paper, we mainly investigate some properties of strongly n -Ding projective and injective modules under the extension of rings, which mainly including excellent extensions, polynomial extension and localizations.

Key words: excellent extensions; strongly n -Ding projective modules; strongly n -Ding injective modules.

1. INTRODUCTION

Throughout the paper R is a commutative ring with identity element, and all R -module are unital. If M is any R -module, we use $pd_R(M)$ and $id_R(M)$ to denote projective and injective dimensions of M .

In [5] and [7], the author introduced strongly Gorenstein flat and Gorenstein FP -injective module, which are defined as follows:

Definition: Let n be a positive integer.

- (1) An R -module M is called strongly Gorenstein flat module (we called Ding projective module) if there is an exact sequence

$$P: \quad \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

Of projective right R -module with $M \cong \text{Ker}(P^0 \rightarrow P^1)$ such that $\text{Hom}_R(-, Q)$ leaves the sequences exact, where Q is a flat R -module.

- (2) An R -module M is called Gorenstein FP -injective module (we called Ding injective module) if there is an exact sequence

$$E: \quad \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

Of injective right R -module with $M \cong \text{Ker}(E^0 \rightarrow E^1)$ such that $\text{Hom}_R(I, -)$ leaves the sequences exact, where I is an FP -injective module.

The main purpose of this paper is to study some properties of strongly n -Ding projective and injective modules under excellent extensions, polynomial extensions and localizations, respectively and we get some interesting results.

2. STRONGLY n -DINGPROJECTIVE AND INJECTIVE MODULES UNDER CHANGE OF RINGS

We begin with the following lemmas.

Lemma 2.1([11]): An R -module M is called strongly Ding projective module if and only if there exists a short exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$, where P is projective module and $\text{Ext}_R^1(M, F) = 0$ for all flat modules F .

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Lemma 2.2([12]): An R -module M is called strongly Ding injective module if and only if there exists a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$, where E is injective module and $\text{Ext}_R^1(Q, M) = 0$ for all FP -injective modules Q .

Definition 2.3: An R -module M is called strongly n -Ding projective module (S-n D- projective module for short) if there exists a short exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ with $\text{pd}_R(P) \leq n$ and $\text{Ext}_R^{n+1}(M, F) = 0$ for all flat modules F .

Definition 2.4: An R -module M is called strongly n -Ding injective module (S-nD- injective module for short) if there exists a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$ with $\text{id}_R(E) \leq n$ and $\text{Ext}_R^{n+1}(Q, M) = 0$ for all FP -injective modules Q .

Proposition 2.5: Let R be a commutative ring and Q a projective R -module. If M is an S-nD-projective R -module, then $M \otimes Q$ is an S-nD-projective R -module.

Proof: Since M is an S-nD-projective R -module, there is a short exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ with $\text{pd}_R(P) \leq n$. Then $0 \rightarrow M \otimes Q \rightarrow P \otimes Q \rightarrow M \otimes Q \rightarrow 0$ and $\text{pd}_R(P \otimes Q) \leq n$ by [9, ch.2, Theorem3]. Let Q' be any flat R -module. Then $\text{Ext}_R^i(M \otimes_R Q, Q') = \text{Hom}(Q, \text{Ext}_R^i(M, Q')) = 0$ by [9, p.256, Lemma9.20] for all $i > n$. Hence $M \otimes Q$ is an S-nD-projective R -module.

Proposition 2.6: Let R and S be equivalent rings via equivalence $F : R\text{-Mod} \rightarrow S\text{-Mod}$ and $G : S\text{-Mod} \rightarrow R\text{-Mod}$. Then

- (1) M is S-nD-projective $R\text{-Mod}$ if and only if $F(M)$ is S-nD-projective $S\text{-Mod}$.
- (2) M is S-nD-injective $R\text{-Mod}$ if and only if $F(M)$ is S-nD-injective $S\text{-Mod}$.

Proof:

(1) Since M is an S-nD-projective R -module, there is the short exact sequence $P : 0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$, where $\text{pd}_R(P) \leq n$. Then $F(P) : 0 \rightarrow F(M) \rightarrow F(P) \rightarrow F(M) \rightarrow 0$ is a exact sequence of $S\text{-Mod}$ with $\text{pd}_R(F(P)) \leq n$. Let Q be any flat $S\text{-Mod}$. Then $\text{Ext}_S^i(F(M), Q) \cong \text{Ext}_R^i(M, G(Q)) = 0$ for all $i > n$. Hence $F(M)$ is S-nD-projective $S\text{-Mod}$. By $GF(M) \cong M$.

(2) By analogy with the proof of (1).

A ring S is said to be an almost excellent extension of a ring R , if the following conditions are satisfied: (1) The ring S is called right R -projective in case for any right S -module M_S with an S -sub module $N_R | M_R$ implies $N_S | M_S$. For example, every $n \times n$ matrix ring $M_n(R)$ is right R -projective. (2) The ring extension $S \geq R$ is called a finite normalization extension in case there is a finite subset $\{s_1, \dots, s_n\}$ of S such that $S = \sum_i^n s_i R$ and $s_i R = R s_i$ for $i = 1, \dots, n$. (3) A finite normalization extension $S \geq R$ is called an excellent extension in case condition (1) is satisfied and ${}_R S, S_R$ are free module with a common as is $\{s_1, \dots, s_n\}$. For example, every $n \times n$ matrix ring $M_n(R)$ is an excellent extension.

Proposition 2.7: Assume that $S \geq R$ is an excellent extension, if ${}_R M$ is a S-nD projective $R\text{-mod}$, then $S \otimes M$ is a S-n D projective $S\text{-mod}$.

Proof: There exists an exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ in $R\text{-mod}$ with $\text{pd}_R(P) \leq n$. Then $0 \rightarrow S \otimes M \rightarrow S \otimes P \rightarrow S \otimes M \rightarrow 0$ is exact in $S\text{-mod}$, with $\text{pd}_R(S \otimes P) \leq n$. Let Q be any flat $S\text{-mod}$,

then Q is a flat R -mod, and so $Ext_S^i(S \otimes M, Q) \cong Ext_R^i(M, Q) = 0$ by [9, p.258, 9.21] for all $i > n$. It follows that $S \otimes M$ is a S-nD projective S -mod.

Proposition 2.8: Let R be a ring and let $n \geq 1$ be a natural number. For any $M \in M_n(R)$ -Mod. Then

- (1) $(1)_R M$ is S-nD projective R -mod if and only if $M_n(R) \otimes M$ is S-nD projective R -mod.
- (2) $(2)_R M$ is S-nD injective R -mod if and only if $Hom_R(M_n(R), M)$ is S-nD injective R -mod.

If R is a ring, then $R[x]$ is the polynomial ring. If M is a left R -module, write $M[x] = R[x] \otimes_R M$. Since $R[x]$ is a free R -module and since tensor product commutes with sums, we may regard the elements of $M[x]$ as 'vectors' $x^i \otimes_R m_i$, $i \geq 0$, $m_i \in M$ with almost all $m_i = 0$.

Proposition 2.9: Let R be a commutative ring. If M is a S-nD projective R -mod, then $M[x]$ is a S-nD projective $R[x]$ -mod.

Proof: There is an exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ in R -mod with $pd_R(P) \leq n$. so $0 \rightarrow M[x] \rightarrow P[x] \rightarrow M[x] \rightarrow 0$ is exact in $R[x]$ -mod and $pd_{R[x]}(P[x]) \leq n$. Let Q be any flat $R[x]$ -mod. Then $Q[x] \cong R[x] \otimes Q \cong R^N \otimes_R Q \cong Q^N$. Hence $Q[x]$ is a flat $R[x]$ -module, and so Q is a flat R -module. Thus $Ext_{R[x]}^i(M[x], Q) = Ext_R^i(M, Q) = 0$ by [9, p.258, 9.21], for all $i > n$, and hence $M[x]$ is an S-nD projective $R[x]$ -module.

Let R be a commutative ring and S a multiplicatively closed set of R . Then $S^{-1}R = R \times S / \sim = \{a/s \mid a \in R, s \in S\}$ is a ring and $S^{-1}M = M \times S / \sim = \{x/s \mid x \in M, s \in S\}$ is an $S^{-1}R$ -module. If P is a prime ideal of R and $S = R - P$, then we will denote $S^{-1}M, S^{-1}R$ by M_P, R_P respectively.

Proposition 2.10: Let R be a commutative ring and S a multiplicatively closed set of R . If M is a S-nD projective R -module, then $S^{-1}M$ is an S-nD projective $S^{-1}R$ -mod.

Proof: Since M is a S-nD projective R -module, there exists an exact sequence $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$ in R -mod with $pd_R(P) \leq n$. Then $0 \rightarrow S^{-1}M \rightarrow S^{-1}P \rightarrow S^{-1}M \rightarrow 0$ is exact in $S^{-1}R$ -mod and $pd_{S^{-1}R}(S^{-1}P) \leq n$. Let \bar{Q} be any flat $S^{-1}R$ -mod, then \bar{Q} is a flat R -module by [11, Lemma 3.20], so

$$Ext_{S^{-1}R}^i(S^{-1}M, \bar{Q}) \cong Ext_{S^{-1}R}^i(S^{-1}R \otimes M, \bar{Q}) \cong Ext_R^i(M, \bar{Q}) = 0 \text{ by [9, p.258, 9.21] for all } i > n,$$

Hence $S^{-1}M$ is an S-nD projective $S^{-1}R$ -mod.

Proposition 2.11: Let R be a commutative ring and S a multiplicatively closed set of R . If $S^{-1}R$ is a projective R -module, then

- (1) If M is an S-nD injective R -module, then $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.
- (2) For any R -module M , $Hom(S^{-1}R, M)$ is an S-n D injective R -module if and only if $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.

Proof: There is an exact sequence $0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0$ in R -mod with $id_R(E) \leq n$. Then $0 \rightarrow Hom(S^{-1}R, M) \rightarrow Hom(S^{-1}R, E) \rightarrow Hom(S^{-1}R, M) \rightarrow 0$ is exact in $S^{-1}R$ -mod and $id_{S^{-1}R}(Hom(S^{-1}R, E)) \leq n$ by [6, Theorem 3.2.9]. Let \bar{I} be any injective $S^{-1}R$ -module. Then \bar{I} is an injective R -module by [4, lemma1.2]. So $Ext_{S^{-1}R}^i(\bar{I}, Hom(S^{-1}R, M)) \cong Ext_R^i(\bar{I}, M) = 0$ by [9, p.258, 9.21] for all $i > n$, and hence $Hom(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module.

(2) is obvious.

Since $\text{Hom}(S^{-1}R, M)$ is an S-n D injective $S^{-1}R$ -module, then there exists an exact sequence $0 \rightarrow \text{Hom}(S^{-1}R, M) \rightarrow \bar{E} \rightarrow \text{Hom}(S^{-1}R, M) \rightarrow 0$ in $S^{-1}R$ -mod with $\text{id}_{S^{-1}R}(\bar{E}) \leq n$. Then \bar{E} is an injective R -module. Let I be any injective R -module. Then $S^{-1}I$ is an injective $S^{-1}R$ -module. So $\text{Ext}_R^i(I, \text{Hom}_{S^{-1}R}(S^{-1}R, \text{Hom}_R(S^{-1}R, M))) \cong \text{Ext}_{S^{-1}R}^i(S^{-1}I, \text{Hom}(S^{-1}R, M)) = 0$ by [8, proposition 5.17] and by [9, p.258,9.21] for all $i > n$, and hence $\text{Hom}(S^{-1}R, M)$ is an S-n D injective R -module.

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