International Research Journal of Pure Algebra-7(4), 2017, 513-521
Available online through www.rjpa.info ISSN 2248–9037

ADDITIVE MAPS PRESERVING DETERMINANT ON MODULES OF MATRICES OVER Z_m

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(Received On: 11-03-17; Revised & Accepted On: 08-04-17)

ABSTRACT

Let $m>1$ be a positive integers, Z_m the residue class ring of module m and $M_2(Z_m)$ the module of all 2×2 matrices over Z_m. For a matrix $A\in M_2(Z_m)$, we denote by $|A|$ the determinant of A. The aim of this paper is to characterize additive maps $\varphi: M_2(Z_m)\to M_2(Z_m)$ such that $|\varphi(A)|=|A|$.

Key words: *Additive maps, preserving determinant, matrix.*

1. INTRODUCTION

Suppose C is the field of all complex numbers, R is a ring and Z is the integral ring. Let $M_n(R)$ be the module of all $n\times n$ matrices over R, and $GL_2(R)$ the subsets of $M_2(R)$ consisting of all invertible matrices. For integers $a,b,c,$ not all 0, denote by (a,b,c) the greatest common divisor of them, and if a divides b, then denote $a|b$. For a matrix $A\in M_n(R)$, we denote by A^t the transpose of A, by A^{-1} the inverse of A, and by $|A|$ the determinant of A. A map $\sigma: M_n(R)\to M_n(R)$ is said to preserve determinant if $|\sigma(A)|=|A|$ for any $A\in M_n(R)$. Denote by E_{ij} the matrix with 1 in the (i,j) th entry and 0 elsewhere. Throughout the paper, $m>1$ is an integer with the prime-power factorization $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$, Z_m is the residue class ring of module m, and φ is an additive determinant preserver on $M_2(Z_m)$.

The problem studied by this paper belongs to preserver problems, which concerns the characterization of the maps on matrices that leave certain function, subsets, relations, etc., invariant. The earliest paper about this problem can be date back to 1897, in [1], Frobenius studied the linear operators preserving determinant on $M_n(C)$. Since then, especially in the last few decades there have been a lot of papers in this kind of problem. It was once one of the most active directions in matrix algebra. The early articles discussed so far are mostly on linear maps on matrices over fields. With the deepening and complexity of research, the problem was naturally transported to general maps or tensor products of matrices or matrices over rings. See [2-11] and their references. The purpose of this paper is to characterize maps preserving determinant on $M_2(Z_m)$. Our main result reads as follow.

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MAIN THEOREM

Suppose $\varphi: M_2(Z_m) \to M_2(Z_m)$ is an additive map, then φ preserves determinant if and only if there exist $P, Q \in GL_2(Z_m)$ such that

$$\varphi\left(\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}\right) = P\begin{bmatrix} \overline{a} & \overline{yb+uc} \\ \overline{ub+yc} & \overline{d} \end{bmatrix}Q, \ \forall \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} \in M_2(Z_m)$$

where $\overline{yu} = \overline{0}$, $\overline{y^2+u^2} = \overline{1}$,and $|PQ| = \overline{1}$

In particular, if $s = 1$, then $\varphi(A) = PAQ$, $\forall A \in M_2(Z_m)$ or $\varphi(A) = PA^tQ, \forall A \in M_2(Z_m)$.

It is easy to see the following corollary holds from the proof of the theorem.

Corollary: Suppose $\phi: M_2(Z) \to M_2(Z)$ is an additive map, then ϕ preserves determinant if and only if there exist $P, Q \in GL_2(Z)$ such that $\phi(A) = PAQ, \ \forall A \in M_2(Z)$ or $\phi(A) = PA^tQ, \forall A \in M_2(Z)$.

2. PRELIMINARY RESULTS

Before proving our main result, let us write four simple lemmas which we will need in the sequel. We start with two well-known statement.

Lemma 1: Suppose a is an integer.
(1) If $(a, m) = 1$, then a is a unit in Z_m;
(2) If $(a, m) \neq 1, m$, then a is zero factor.

Lemma 2: Let a, b and c be nonzero integers. Then the equation $ax + by + cz = 1$ has integral solutions if and only if $(a, b, c) = 1$.

Lemma 3: φ is linear.

Proof: From $\varphi(\overline{0}) = \varphi(\overline{0} + \overline{0}) = \varphi(\overline{0}) + \varphi(\overline{0})$, we know $\varphi(\overline{0}) = O$.

For any $\overline{k} \in Z_m$ with $0 < k < m,$ let $A \in M_2(Z_m)$, then

$$\varphi(\overline{k}A) = \varphi(kA) = k\varphi(A) = \overline{k}\varphi(A).$$

Lemma 4: φ is injective.

Proof: Suppose $\varphi(A) = \varphi(B)$, then $\varphi(A - B) = O.$ If $A - B \neq O$, then there exist $P, Q \in GL_2(Z_m)$ such that

$$A - B = P\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix}Q$$

with $\overline{a} \neq \overline{0}.$ Since $|PQ|\overline{ad-bc} = |A - B| = |\varphi(A - B)| = |O| = \overline{0}$, $\overline{ad-bc} = \overline{0}.$ Set $C = PE_{22}Q$, then

$$|PQ|\overline{a} = |A - B + C| = |\varphi(A - B + C)| = |\varphi(A - B) + \varphi(C)| = |\varphi(C)| = \overline{0},$$

thus $\overline{a} = \overline{0}$, it is a contradiction. Hence $A - B = O,\ i.e., A = B,\ \varphi$ is injective.

Lemma 5: If $A=\begin{bmatrix}\overline{a_1} & \overline{b_1}\\ \overline{c_1} & \overline{d_1}\end{bmatrix}$, where $0\le a,b,c,d<m,$ then there exist $P,Q\in GL_2\left(Z_m\right)$ such that $PAQ=\begin{bmatrix}\overline{a_2} & \overline{b_2}\\ \overline{c_2} & \overline{d_2}\end{bmatrix}$, where $\overline{a_2}$ divides $\overline{b_2},\overline{c_2}$ and $\overline{d_2}$.

Proof: If $\overline{a_1},\ \overline{b_1},\ \overline{c_1},\ \overline{d_1}$ are all $\overline{0}$, then the conclusion is natural, In the following, we consider the case: there is one which is not equal to $\overline{0}$ in $\overline{a_1},\ \overline{b_1},\ \overline{c_1},\ \overline{d_1}$. Without loss of generality, we may suppose $\overline{a_1}\neq\overline{0}$.

(1) If $\overline{a_1}$ does not divide $\overline{b_1}$, then a_1 does not divide b_1 Suppose $b_1=q_1a_1+r_1,$ where q_1 and r_1 are integers, and $0<r_1<a_1.$ Set

$$Q_1=\begin{bmatrix}\overline{0} & \overline{1}\\ \overline{1} & \overline{0}\end{bmatrix}\begin{bmatrix}\overline{1} & \overline{q_1}\\ \overline{0} & \overline{1}\end{bmatrix}Q$$

then

$$A=P\begin{bmatrix}\overline{r_1} & *\\ * & *\end{bmatrix}Q_1, \text{ where } 0<r_1<a_1<m.$$

(2) If $\overline{a_1}$ does not divide $\overline{c_1}$,using a similar way to (1), we deduce that there exists $P_1\in GL_2\left(Z_m\right)$ such that

$$A=P_1\begin{bmatrix}\overline{r_2} & *\\ * & *\end{bmatrix}Q_1, \text{ where } 0<r_2<a_1<m.$$

(3) Else if $\overline{a_1}\big|\overline{b_1},\overline{a_1}\big|\overline{c_1},\overline{a_1}$ does not divide $\overline{d_1}$. Suppose $\overline{c_1}=\overline{q_2a_1}.$ Set

$$P_2=P\begin{bmatrix}\overline{1} & \overline{0}\\ \overline{q_2} & \overline{1}\end{bmatrix},\ Q_2=\begin{bmatrix}\overline{1} & \overline{-1}\\ \overline{0} & \overline{1}\end{bmatrix}Q,$$

then

$$A=P_2\begin{bmatrix}\overline{a_1} & \overline{d_1+\left(1-q_2\right)b_1}\\ \overline{0} & \overline{d_1-q_2b_1}\end{bmatrix}Q_2.$$

Because $\overline{a_1}\big|\overline{b_1}$, but $\overline{a_1}$ does not divide $\overline{d_1}$, so $\overline{a_1}$ does not divide $\overline{d_1+\left(1-q_2\right)b_1}.$ Using a similar way to (1), we deduce that there exists $Q_3\in GL_2\left(Z_m\right)$ such that

$$A=P_2\begin{bmatrix}\overline{r_3} & *\\ * & *\end{bmatrix}Q_3,\ \text{where } 0<r_3<a_1<m.$$

Combining (1),(2) and (3), if there is one which is not divided by $\overline{a_1}$ in $\overline{b_1},\overline{c_1},\overline{d_1}$, then there exist $P_4,Q_4\in GL_2\left(Z_m\right)$ such that

$$A=P_4\begin{bmatrix}\overline{l_1} & \overline{l_2}\\ \overline{l_3} & \overline{l_4}\end{bmatrix}Q_4,\ \text{where } 0<l_1<a_1<m,\ 0<l_2,l_3,l_4<m.$$

If there is one which is not divided by $\overline{l_1}$ in $\overline{l_2},\overline{l_3},\overline{l_4},\cdots$, then there exist P_5 , $Q_5\in GL_2\left(Z_m\right)$ such that

$$A=P_5\begin{bmatrix}\overline{v_1} & \overline{v_2}\\ \overline{v_3} & \overline{v_4}\end{bmatrix}Q_5,\ \text{where } 0<v_1<l_1<m,\ 0<v_2,v_3,v_4<m.$$

Because a_1 is a finite positive integer, so the above process must stop after finite steps. Thus there exist P_6, $Q_6 \in GL_2(Z_m)$ such that

$$A = P_6 \begin{bmatrix} \overline{a_2} & \overline{b_2} \\ \overline{c_2} & \overline{d_2} \end{bmatrix} Q_6$$

where $0 < a_2 < a_1 < m,\ 0 \le b_2, c_2, d_2 < m,$ and $\overline{a_2} \big| \overline{b_2}$, $\overline{a_2} \big| \overline{c_2}$, $\overline{a_2} \big| \overline{d_2}$.

3. PROOF OF THE MAIN THEOREM

Since the sufficiency part of the Main Theorem is easy, we omit it. In the following we consider only the necessity. We divide the proof into three steps.

Step-1: There exist $P, Q \in GL_2(Z_m)$ such that $\varphi(E_{11}) = PE_{11}Q$.

Note that φ is injective, by lemma 5 we know that there exist $P_1, Q_1 \in GL_2(Z_m)$ such that $\varphi(E_{11}) = P_1 \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} Q_1$,

where $0 < a < m$, and $\overline{a}$ divides $\overline{b}$, $\overline{c}$ and $\overline{d}$.

Suppose $\overline{b} = \overline{q_1 a}$, $\overline{c} = \overline{q_2 a}$, $\overline{d} = \overline{q_3 a}$. Set

$$P_2 = P_1 \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{q_2} & \overline{1} \end{pmatrix},\ Q = \begin{pmatrix} \overline{1} & \overline{q_1} \\ \overline{0} & \overline{1} \end{pmatrix} Q_1,$$

then

$$\varphi(E_{11}) = P_2 \begin{pmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{qa} \end{pmatrix} Q,$$

where $\overline{q} = \overline{q_3 - q_1 q_2}$.

If $\overline{a}$ is zero divisor, i.e., there exists an integer a_1 such that $0 < a_1 < m$ and $\overline{a_1 a_2} = \overline{0}$, then

$$\varphi\left(\overline{a_1} E_{11}\right) = \overline{a_1} \varphi\left(E_{11}\right) = P_2 \begin{bmatrix} \overline{a_1 a} & \overline{0} \\ \overline{0} & \overline{q a_1 a} \end{bmatrix} Q = 0.$$

It is contradict to the injectivity of φ. Thus $\overline{a}$ is a unit. Since

$$\overline{0} = \left|E_{11}\right| = \left|\varphi\left(E_{11}\right)\right| = \left|P_2\right|\left|Q\right|\overline{qa^2},$$

$\overline{q} = \overline{0}.$ Set

$$P = P_2 \begin{bmatrix} \overline{a} & \overline{0} \\ \overline{0} & \overline{1} \end{bmatrix},$$

then $\varphi\left(E_{11}\right) = PE_{11}Q.$

Step-2: There exist $M, N \in GL_2(Z_m)$ such that $\varphi\left(E_{ii}\right) = ME_{ii}N, i = 1, 2.$

Suppose $\varphi(E_{22}) = P \begin{pmatrix} \overline{x} & \overline{y} \\ \overline{u} & \overline{v} \end{pmatrix} Q$. Since

$$\overline{0} = \left|E_{22}\right| = \left|\varphi(E_{22})\right| = \left|P\right|\left|Q\right|\overline{xv - yu},\ \overline{xv - yu} = \overline{0}.$$

Moreover

$$\bar{1} = |E_{11}+E_{22}| = |\varphi(E_{11}+E_{22})| = |\varphi(E_{11})+\varphi(E_{22})|$$
$$= \left| P\begin{bmatrix} \overline{x+1} & \bar{y} \\ \bar{u} & \bar{v} \end{bmatrix} Q \right| = |PQ|\overline{(x+1)v-yu}.$$

Thus $|PQ|\bar{v} = \bar{1}$ is a unit. Set

$$M = P\begin{bmatrix} \bar{1} & \bar{v}^{-1}\bar{y} \\ \bar{0} & \bar{1} \end{bmatrix},\ N = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{v} \end{bmatrix}\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{v}^{-1}\bar{u} & \bar{1} \end{bmatrix}Q,$$

then

$$\varphi(E_{11}) = ME_{11}N,\ \varphi(E_{22}) = M\begin{bmatrix} \overline{x_1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}N,$$

where $\overline{x_1} = \bar{x} - \bar{v}^{-1}\overline{uy}$. Finally, using $|\varphi(E_{22})| = |E_{22}| = \bar{0}$ we get $\overline{x_1} = \bar{0}$.

Step-3: $\varphi(E_{ii}) = ME_{ii}N, i = 1, 2.$

$$\varphi(E_{12}) = M\begin{bmatrix} \bar{0} & \bar{y} \\ \bar{u} & \bar{0} \end{bmatrix}N,\ \varphi(E_{21}) = M\begin{bmatrix} \bar{0} & \bar{u} \\ \bar{y} & \bar{0} \end{bmatrix}N,$$

Suppose $\varphi(E_{12}) = M\begin{bmatrix} \overline{x_1} & \overline{y_1} \\ \overline{u_1} & \overline{v_1} \end{bmatrix}Q$. Since

$$\bar{0} = |E_{12}| = |\varphi(E_{12})| = |MN|\overline{x_1v_1 - y_1u_1},$$
$$\overline{x_1v_1 - y_1u_1} = \bar{0} \tag{1}$$

Moreover,

$$\bar{0} = |E_{11}+E_{12}| = |\varphi(E_{11}+E_{12})| = |\varphi(E_{11})+\varphi(E_{12})|$$
$$= \left| M\begin{bmatrix} \overline{x_1+1} & \bar{y} \\ \bar{u} & \bar{v} \end{bmatrix} N \right| = |MN|\overline{(x_1+1)v_1 - y_1u_1},$$

thus $\overline{v_1} = \bar{0}$. If we consider $\varphi(E_{22}+E_{12})$, then we get $\overline{x_1} = \bar{0}$. Hence

$$\varphi(E_{12}) = P\begin{bmatrix} \bar{0} & \overline{y_1} \\ \overline{u_1} & \bar{0} \end{bmatrix}Q,\ \overline{y_1u_1} = \bar{0}. \tag{2}$$

Similarly,

$$\varphi(E_{21}) = P\begin{bmatrix} \bar{0} & \overline{y_2} \\ \overline{u_2} & \bar{0} \end{bmatrix}Q,\ \overline{y_2u_2} = \bar{0}. \tag{3}$$

Meanwhile, since

$$\bar{0} = |E_{11}+E_{12}+E_{21}+E_{22}| = |\varphi(E_{11}+E_{12}+E_{21}+E_{22})|$$
$$= \left| M\begin{bmatrix} \bar{1} & \overline{y_1+y_2} \\ \overline{u_1+u_2} & \bar{1} \end{bmatrix} N \right| = |MN|\overline{(y_1+y_2)(u_1+u_2)-1},\ \overline{(y_1+y_2)(u_1+u_2)} = \bar{1},$$

this together with Eqs.(2) and (3) implies

$$\overline{y_2u_1 + y_1u_2} = \bar{1} \tag{4}$$

Set $x = y_1 + y_2$, then $\bar{x}$ is a unit of Z_m. Using Eps. $(2)-(4)$ we have

$$\overline{x^2u_2} = \overline{(y_1+y_2)^2 u_2} = \overline{y_1^{\,2}u_2} = \overline{(1-y_2u_1)\,y_1} = \overline{y_1},$$

i.e., $\overline{xu_2} = \bar{x}^{-1}\overline{y_1}$. Similarly, $\overline{xu_1} = \bar{x}^{-1}\overline{y_2}$. Denote $\bar{y} = \overline{xu_2}$, $\bar{u} = \overline{xu_1}$. Set

$$M_3 = M\begin{bmatrix} \bar{1} & \\ & \bar{x}^{-1} \end{bmatrix}, N_3 = \begin{bmatrix} \bar{1} & \\ & \bar{x} \end{bmatrix} N,$$

$$\varphi\left(\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}\right) = M_3\begin{bmatrix} \bar{a} & \overline{yb+uc} \\ \overline{ub+yc} & \bar{d} \end{bmatrix} N_3, \forall \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \in M_2(Z_m),$$

where $\bar{y}, \bar{u} \in Z_m$, and

$$\overline{yu} = \bar{0},\ \overline{y^2+u^2} = \bar{1}. \tag{5}$$

Now we discuss in three cases.

Case-1: If $\bar{u} = \bar{0}$, then $\overline{y^2} = \bar{1}$, thus $\bar{y}$ is a unit. Set

$$M_1 = M\begin{bmatrix} \bar{1} & \\ & \bar{y} \end{bmatrix},\ N_1 = \begin{bmatrix} \bar{1} & \\ & \bar{y} \end{bmatrix} N,$$

then

$$\varphi(E_{ij}) = M_1 E_{ij} N_1,\ i, j = 1, 2. \tag{6}$$

Case-2: If $\bar{u}$ is a unit, then $\bar{y} = \bar{0}$. Set

$$M_2 = M\begin{bmatrix} \bar{1} & \\ & \bar{u} \end{bmatrix}, N_2 = \begin{bmatrix} \bar{1} & \\ & \bar{u} \end{bmatrix} N$$

then

$$\varphi(E_{ij}) = M_2 E_{ji} N_2,\ i, j = 1, 2. \tag{7}$$

Case-3: Else if $\bar{u}$ is neither zero unit, then from Case 1 and 2 we know $\bar{y}$ is also neither zero nor unit.

(1) Ifs=1, i.e., $m = p_1^{\alpha_1}$, then $p_1 | u, y$. By Eq. (5) we have an integer k such that $y^2 + u^2 = km + 1$. Thus $p_1 | 1$, it is a contradiction.

(2) Else if $s > 1$. then by Eq. (5) we know $m | yu$, therefore $p_k | yu$ for any $k \in \{1, 2, \cdots, s\}$, which implies $p_k | y$ or $p_k | u$. If $p_k | y$ and $p_k | u$, then we get a contradiction in a similar way to (1). Hence, p_k divide only one of y and u. Without loss of generality, we suppose

$$y = p_1^{\beta_1} \cdots p_t^{\beta_t} y_3,\ u = p_{t+1}^{\beta_{t+1}} \cdots p_s^{\beta_s} u_3$$

where $1 \le t < s$, $(y_3, m) = (u_3, m) = 1$, $\alpha_j \le \beta_j$, $j = 1, \cdots s$ .Because

$$\left(p_1^{\beta_1} \cdots p_t^{\beta_t},\ p_{t+1}^{\beta_{t+1}} \cdots p_s^{\beta_s},\ m\right) = 1,$$

by Lemma 4, the equation

$$p_1^{\beta_1} \cdots p_t^{\beta_t} e + p_{t+1}^{\beta_{t+1}} \cdots p_s^{\beta_s} f + mg = 1 \tag{8}$$

has integral solutions, i.e., there are integers e_0, f_0 and g_0 such that $p_1^{\beta_1}\cdots p_t^{\beta_t}e_0+p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}f_0+mg_0=1$. Set $y_3=e_0, u_3=f_0$, then

$$\overline{y^2+u^2}=\overline{\left(p_1^{\beta_1}\cdots p_t^{\beta_t}e_0\right)^2+\left(p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}f_0\right)^2}=\overline{\left(p_1^{\beta_1}\cdots p_t^{\beta_t}e_0+p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}f_0\right)^2}=\bar{1}.$$

Moreover, $\overline{yu}=\bar{0}$ is obviously. Thus, there are indeed $\bar{y}$ and $\bar{u}$ which is neither zero nor unit but satisfy Eq. (5).

The Main Theorem is proved completely.

4. NOTE

Note-1: From the proof of the theorem, if $s=1$, then the form of φ can be (6) or (7). Else if $s>1$ besides (6) and (7), φ has other forms. Because the equation (8) has infinitely many solutions, so there are many $\bar{y}$ and $\bar{u}$ to satisfy Eq. (5), which depends on m. Generally, we have

Proposition 1: Suppose φ_1 and φ_2 are additive maps preserving determinant on $M_2\left(Z_m\right)$.
If

$$\varphi_1\left(\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\right)=P_1\begin{bmatrix}\bar{a} & \overline{yb+uc}\\ \overline{ub+yc} & \bar{d}\end{bmatrix}Q_1,\ \overline{yu}=\bar{0},\ \overline{y^2+u^2}=\bar{1},$$

$$\varphi_2\left(\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\right)=P_2\begin{bmatrix}\bar{a} & \overline{rb+hc}\\ \overline{hb+rc} & \bar{d}\end{bmatrix}Q_2,\ \overline{rh}=\bar{0},\ \overline{r^2+h^2}=\bar{1},$$

then there exist $M,N\in GL_2\left(Z_m\right)$ such that

$$\varphi_1\left(A\right)=M\varphi_2\left(A\right)N,\ \forall A\in M_2\left(Z_m\right)$$

if and only if then there exists a $\bar{\omega}$ unit in Z_m such that $\overline{\omega^2}=\bar{1}$, and

$$\bar{y}=\overline{\omega r},\ \bar{u}=\overline{\omega h},$$

where $P_i,Q_i\in GL_2\left(Z_m\right), i=1,2$.

Proof: Firstly, we prove the sufficiency.

Set

$$M=P_1\begin{bmatrix}\bar{1} & \\ & \bar{\omega}\end{bmatrix}P_2^{-1},\ N=Q_2^{-1}\begin{bmatrix}\bar{1} & \\ & \bar{\omega}\end{bmatrix}Q_1$$

then

$$\varphi_1\left(A\right)=P_1\begin{bmatrix}\bar{x}_1 & \bar{x}_2\\ \bar{x}_3 & \bar{x}_4\end{bmatrix}P_2^{-1},\ \varphi_2\left(A\right)=Q_2^{-1}\begin{bmatrix}\bar{v}_1 & \bar{v}_2\\ \bar{v}_3 & \bar{v}_4\end{bmatrix}^{-1}Q_1,\ \forall A\in M_2\left(Z_m\right).$$

Substituting $A=E_{ij}, i,j=1,2$ into the above formula respectively, we have

$$M=P_1\begin{bmatrix}\bar{x}_1 & 0\\ 0 & \bar{x}_4\end{bmatrix}P_2^{-1},\ \bar{y}=\bar{x}_4^{-1}\overline{x_1 r},\ \bar{h}=\bar{x}_4^{-1}\overline{x_1 u}.$$

Denote $\bar{\omega}=\bar{x}_4^{-1}\bar{x}_1$, then $\bar{y}=\overline{\omega r}$, $\bar{h}=\overline{\omega u}$. Since $M\in GL_2\left(Z_m\right)$, both $\bar{x}_1$ and $\bar{x}_4$ are unit, therefore $\bar{\omega}$ is a unit.
Because $\overline{r^2+h^2}=\bar{1}$, so $\overline{\omega^2}^{-1}\overline{y^2}+\overline{\omega^2}^{-1}\overline{y^2}+\overline{\omega^2u^2}=\bar{1}$, i.e.,

$$\overline{y^2}+\overline{\omega^4u^2}=\overline{\omega^2}. \tag{9}$$

Substituting $\overline{y^2}=\overline{1-u^2}$ into the above formula, we have $\overline{1-u^2+\omega^4u^2}=\overline{\omega^2}$, i.e., $\overline{\left(\omega^4-1\right)u^2}=\overline{\omega^2-1}$. Similar to proof of theorem, without loss of generality, we may suppose

$$y=p_1^{\beta_1}\cdots p_t^{\beta_t}y_3,\ u=p_{t+1}^{\beta_{t+1}}\cdots p_s^{\beta_s}u_3 \tag{10}$$

where $0\le t\le s$, $\left(y_3,m\right)=\left(u_3,m\right)=1$, $\alpha_j\le\beta_j$, $j=1,\cdots s$. Then,

$$\overline{\left(\omega^4-1\right)p_{t+1}^{2\beta_{t+1}}\cdots p_s^{2\beta_s}u_3^2}=\overline{\omega^2-1}.$$

Thus,

$$p_{t+1}^{\alpha_{t+1}}\cdots p_s^{\alpha_s}\left|\left(\omega^2-1\right)\right. \tag{11}$$

Similarly, substituting $\overline{u^2}=\overline{1-y^2}$ into Eq. (9) will bring out

$$p_1^{\alpha_1}\cdots p_t^{\alpha_t}\left|\left(1-\omega^2\right)\right. \tag{12}$$

Combining Eqs. (11) and (12), we get $m\left|\left(\omega^2-1\right)\right.$, i.e., $\overline{\omega^2}=\bar{1}$, thus $\bar{\omega}$ is a unit and $\bar{y}=\overline{\omega r}$, $\bar{u}=\overline{\omega h}$.

Note-2: We call φ_1 and φ_2 of proposition 1 are the same class,

Example: Suppose $P,Q\in GL_2\left(Z_{420}\right)$, and $\left|PQ\right|=1$. For any $\bar{a},\bar{b},\bar{c},\bar{d}\in Z_{420}$, Set

$$\sigma_1\left(\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\right)=P\begin{bmatrix}\bar{a} & \overline{196b+225c}\\ \overline{225b+196c} & \bar{d}\end{bmatrix}Q,$$

$$\sigma_2\left(\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\right)=P\begin{bmatrix}\bar{a} & \overline{56b-15c}\\ \overline{-15b+56c} & \bar{d}\end{bmatrix}Q,$$

$$\sigma_3\left(\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix}\right)=P\begin{bmatrix}\bar{a} & \overline{36b-35c}\\ \overline{-35b+36c} & \bar{d}\end{bmatrix}Q,$$

then $\sigma_1,\sigma_2,\sigma_3$ are all linear maps preserving determinant on $M_2\left(Z_{70}\right)$.

It is clear that $\overline{41}$ is a unit in Z_{420}, and

$$\overline{41}^2=\bar{1},\ \overline{196}=\overline{56}\times\overline{41},\ \overline{225}=\overline{\left(-15\right)\times41}.$$

Then, by proposition 1, σ_1 and σ_2 are the same class.

In addition, if there exists a unit $\bar{\omega}\in Z_{420}$ such that $\overline{36\omega}=\overline{196}$, then there exists an integer k such that $196=36\omega+420k$, i.e., $2^2\times7^2=2^2\times3^2\omega+2^2\times3\times5\times7k$, therefore $7\left|\omega\right.$. But $\bar{\omega}$ is a unit, so $\left(\omega,420\right)=1$, it is a contradiction. Hence, by proposition1, σ_3 and σ_1 are not the same class.

ACKNOWLEDGEMENTS

This work is supported by Education Department of Heilongjiang province of China (NO.12541605) and the National Natural Science Foundation Grants of China (Grant No.11371109).

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Source of Support: Nil, Conflict of interest: None Declared