



**BOUNDEDNESS OF HIGHER ORDER COMMUTATORS
FOR THE PARAMETRIC MARCINKIEWICZ INTEGRAL ON HARDY SPACE**

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ABSTRACT

The authors study the commutators and higher order commutators of para-metric Marcinkiewicz integrals and obtain that the operators bounded from $H_b^1(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with the symbol $b \in BMO$.

Keywords: Hardy space, commutators, parametric Marcinkiewicz integral.

1. INTRODUCTION

Let T be a Calderon-Zygmund singular integral operator. For a locally integral function b on \mathbb{R}^n ; the commutator $[b, T]$ of T is defined by

$$[b, T]f(x) = b(x)T(f)(x) - T(b)(x)f(x).$$

It is well known that the Calderon-Zygmund singular integral operator T is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. However, it was observed in [7] that the corresponding result for $[b, T]$ is false when b is a BMO function. In 1995, Perez in [7] introduced a subspace $H_b^1(\mathbb{R}^n)$; and proved that $[b, T]$ is bounded operator from $H_b^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for $b \in BMO$. Following the definition of [8], in 1998, Alvarez defined the atomic space $H_b^p(\mathbb{R}^n)$, for $0 < p < 1$, and proved that the commutator $[b, T]$ is also bounded from $H_b^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $n/(n+1) < p \leq 1$ in [9].

Before stating our results, let us first give the definitions and some known results on μ_ρ and $\mu_\rho^{b_m}$. Suppose that S^{n-1} is the unit sphere in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be a homogeneous function of degree zero on \mathbb{R}^n satisfying $\Omega \in L^1(S^{n-1})$ and the following property

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0 \tag{1}$$

In 1960, Hörmander in [1] introduced the parametric Marcinkiewicz integrals;

$$\mu_\rho(f)(x) = \left(\int_0^\infty |F_{\rho,t}(x)|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}}$$

where $0 < \rho < n$ and

$$F_{\rho,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy$$

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If $\rho=1$, then is the classical Marcinkiewicz integrals defined by Stein in [4]. Stein proved that if Ω is continuous and satisfies a Lip_α ($0 < \alpha \leq 1$) condition on S^{n-1} , then is of type (p,p) for $1 < p \leq 2$, and of weak type (1,1). On the other hand, let $b \in BMO(R^n)$ and $m \in N$, the m order commutators generated by the parametric Marcinkiewicz integrals μ_ρ and b is defined by

$$\mu_\rho^{b^m} f(x) = \left(\int_0^\infty |F_{\rho,t}^{b^m}(f)(x)|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}}$$

Where

$$F_{\rho,t}^{b^m}(f)(x) = \int_{|x-y| \leq t} [b(x) - b(y)]^m \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy$$

In [2], Ding, Lu and Xue obtained boundedness of commutators for the Marcinkiewicz integrals on Hardy space. In [3], Shi and Jiang obtained the weighted L^p boundedness of μ_ρ and $\mu_\rho^{b^m}$ with rough kernels. In this paper, we will consider the boundedness of the higher order commutator $\mu_\rho^{b^m}$ on the atomic-Hardy spaces $H_{b^m}^1(R^n)$.

To state our results we need some notations and definitions. A function $\Omega(x')$ on S^{n-1} is said to satisfy the L^q -Dini condition $q > 1$, if $\Omega(x') \in L^q(S^{n-1})$ and $\int_0^1 \frac{\omega_q(\sigma)}{\sigma} d\sigma < \infty$ (2)

Where $\omega(\sigma) = \sup_{|\rho| \leq \sigma} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{\frac{1}{q}}$

and is a rotation on S^{n-1} , $|\rho| = \|\rho - I\|$, We say that $\Omega \in Lip(S^{n-1})$ for $0 < \alpha \leq 1$.

If there exists a constant $M > 0$ such that for any

$$x, y \in S^{n-1}, |\Omega(x) - \Omega(y)| \leq M |x - y|^\alpha.$$

Definition 1: Let $b \in BMO(R^n)$, $m \in N$ and $1 \leq t \leq \infty$. A function $a(x)$ is said to be a $(1, t, b^m)$ -atoms, if it satisfies the following conditions:

- (1) $\text{supp } a \subset Q(x_0, r)$;
- (2) $\|a\|_t \leq Q^{1/t-1}$;
- (3) $\int_{R^n} a(x) dx = \int_{R^n} a(x)b(x) dx = \dots = \int_{R^n} a(x)b^m(x) dx = 0$,

Where Q is a ball in R^n .

Now let us state our main results.

Theorem 1: Let Ω is a homogeneous function of degree zero on R^n satisfying (1) and the L^q -Dini condition for $q > 1$, Then there exists a constant $C > 0$, independent of f such that $\|\mu_\rho^{b^m}(f)\|_{L^1} \leq C \|f\|_{H_{b^m}^1}$.

2. PROOF OF THEOREMS

Because the proof of the following Lemma is very similar to Lemma 5 in [6], we just formulate it.

Lemma 1: Suppose that is homogeneous of degree zero and satisfies L^q -Dini condition for $q > 1$, If there exist a constant $\alpha_0, 0 < \alpha_0 < 1/2$, such that $|x| < \alpha_0 R$, then

$$\left(\int_{R < |y| < 2R} \left| \frac{\Omega(x-y)}{|y-x|^{n-\rho}} - \frac{\Omega(y)}{|y|^{n-\rho}} \right|^q dy \right)^{\frac{1}{q}} \leq CR^{n/q-(n-1)} \left(\frac{|x|}{R} + \int_{|x|/2R < \delta < \pi|x|/R} \frac{\omega_q(\delta)}{\delta} d\delta \right),$$

where the constant $C > 0$ is independent of R and x .

When $\omega \equiv 1$, from [3], we know the following conclusion immediately.

Lemma 2: Suppose that Ω be a homogeneous function of degree zero on R^n satisfying (1) and $0 < \rho < n$. If $\Omega \in L^q(S^{n-1})(q > 1)$ and $m \in N, b(x) \in BMO(R^n)$.

Then for $1 < p < \infty$, there exists a constant $C > 0$, independent of f , such that $\|\mu_\rho^{b^m}(f)\|_p \leq C \|f\|_p$.

Now, let us turn to the proof of Theorem 1 only for the case $m = 1$, When $m > 1$, the proof is similar to the case $m = 1$, but more complex in the form and we omit the details by brevity.

Proof of Theorem 1: It suffices to prove that $\|\mu_\rho(a)(x)\|_L \leq C$ for any $(1, t, b)$ -atom $a(x)(1 < t < \infty)$. Fix a ball

$Q = Q(x_0, r) \subset R^n$, and let $Q^* = 8n^{\frac{1}{2}}Q$, we have

$$\int_{R^n} \mu_\rho^b(a)(x) dx = \int_{Q^*} \mu_\rho^b(a)(x) dx + \int_{(Q^*)^c} \mu_\rho^b(a)(x) dx.$$

By Lemma 2, Hölder's inequality and size condition of $a(x)$, we get

$$\begin{aligned} \int_{Q^*} \mu_\rho^b(a)(x) dx &\leq |Q^*|^{1/t'} \left(\int_{Q^*} |\mu_\rho^b(a)(x)|^t dx \right)^{1/t} \\ &\leq |Q|^{1/t'} \left(\int_{R^n} |a(x)|^t dx \right)^{1/t} \leq |Q|^{1/t'} |Q|^{1/t-1} \leq C. \end{aligned}$$

Where $\frac{1}{t} + \frac{1}{t'} = 1$. It remains to show that

$$I := \int_{(Q^*)^c} \mu_\rho(a)(x) dx \leq C.$$

Taking that $(a+b)^l \leq a^l + b^l$, when $a, b \geq 0$ and $0 < l \leq 1$, we have

$$\begin{aligned} I &\leq \int_{(Q^*)^c} \left(\int_0^{|x-x_0|+2r} \left| \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} dx \\ &\quad + \int_{(Q^*)^c} \left(\int_{|x-x_0|+2r}^\infty \left| \int_{|x-y|\leq t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} dx \end{aligned}$$

Since $|x-y| \sim |x-x_0| \sim |x-x_0| + 2r$ for any $x \in Q^*$ and $y \in Q$, we have

$$\left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_0|+2r)^{2\rho}} \right| \leq C \frac{r}{|x-y|^{2\rho+1}}. \quad (3)$$

By (3), we have

$$\begin{aligned} I_1 &\leq \int_{(Q^*)^c} \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |b(x)-b(y)| |a(y)| \left(\int_{|x-y|\leq t \leq |x-x_0|+2r} \frac{dt}{t^{2\rho+1}} \right)^{1/2} dy dx \\ &\leq \int_{(Q^*)^c} \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |b(x)-b(y)| |a(y)| \left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_0|+2r)^{2\rho}} \right|^{1/2} dy dx \\ &\leq Cd^{1/2} \int_Q \int_{(Q^*)^c} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} |b(x)-b(y)| |a(y)| dy \end{aligned}$$

$$\begin{aligned} &\leq Cd^{1/2} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |x-y| \leq 2^{j+1} d} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} |b(x) - C_1| dx |a(y)| dy \\ &+ Cd^{1/2} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |x-y| \leq 2^{j+1} d} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} |b(y) - C_1| dx |a(y)| dy \\ &= I_{1,1} + I_{1,2}, \end{aligned}$$

Where $C_1 = \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} |b(x)| dx$ and $Q_{j+1} = \{x : |x-y| < 2^{j+1} d\}$. Below we give the estimates of $I_{1,1}$ and $I_{1,2}$.

Using Hölder's inequality and the size condition of $a(x)$, We get

$$\begin{aligned} I_{1,1} &\leq Cd^{1/2} \int_Q \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |x-y| \leq 2^{j+1} d} \frac{|Q(x-y)|^q}{|x-y|^{n+1/2}} dx \right)^{1/q} \times \left(\frac{|Q_{j+1}|}{|Q_{j+1}|} \int_{2^j d \leq |x-y| \leq 2^{j+1} d} \frac{|b(x) - C_1|^{q'}}{|x-y|^{n+1/2}} dx \right)^{1/q'} |a(y)| dy \\ &\leq Cd^{1/2} \sum_{j=1}^{\infty} \frac{(2^{j+1} d)^{n/q} (2^{j+1} d)^{n/q'}}{(2^j d)^{(n+1/2)/q} (2^j d)^{(n+1/2)/q'}} \|b\|_{BMO} \int_Q |a(y)| dy \\ &\leq C \sum_{j=1}^{\infty} \frac{1}{(2^j)^{1/2}} \|b\|_{BMO} \int_Q |a(y)| dy \\ &\leq C \|b\|_{BMO} |Q|^{1/t'} \left(\int_Q |a(y)|^t dy \right)^{1/t} \leq C. \end{aligned}$$

On the other hand, note that

$$\int_{2^j d \leq |x-y| \leq 2^{j+1} d} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} dx \leq C \left(\frac{1}{(2^j)^{1/2}} \right).$$

SO

$$\begin{aligned} I_{1,2} &\leq Cd^{1/2} \int_Q \left(\sum_{j=1}^{\infty} \frac{1}{(2^j)^{1/2}} \right) |b(y) - C_1| |a(y)| dy \\ &\leq C \left(\int_Q |b(y) - C_1|^{t'} dy \right)^{1/t'} \left(\int_Q |a(y)|^t dy \right)^{1/t} \\ &\leq C. \end{aligned}$$

For I_2 , using the vanishing condition of $a(x)$, we get

$$\begin{aligned} I_2 &= \int_{(Q^*)^c} \left(\int_{|x-x_0|+2r}^{\infty} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y) dy - \int_{|x-y| \leq t} \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} (b(x) - b(y)) a(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}} dx \\ &\leq C \int_{(Q^*)^c} \int_{R^n} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right| \times |b(x) - b(y)| |a(y)| \left(\int_{|x-x_0|+2r}^{\infty} \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}} dy dx \\ &\leq C \int_{(Q^*)^c} \int_{R^n} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right| \frac{|b(x) - b(y)| |a(y)|}{|x-x_0|} dy dx \end{aligned}$$

$$\begin{aligned} &\leq C \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |x-x_0| \leq 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right| |b(x) - C_2| dx |a(y)| dy \\ &+ C \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |x-x_0| \leq 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right| |b(x) - C_2| dx |a(y)| dy \\ &= I_{2,1} + I_{2,2}. \end{aligned}$$

Where $C_3 = \frac{1}{|Q_{j+1}^1|} \int_{Q_{j+1}^1} |b(x)| dx$ and $Q_{j+1}^1 = \{x : |x-x_0| < 2^{j+1} d\}$. By Hölder's

Inequality, Lemma 1 and the size condition of $a(x)$, we have

$$\begin{aligned} I_{2,1} &\leq C \int_Q \sum_{j=1}^{\infty} \frac{1}{2^j d} \left(\int_{2^j d \leq |x-x_0| \leq 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right|^q dx \right)^{1/q} \\ &\quad \times \left(\int_{2^j d \leq |x-x_0| \leq 2^{j+1} d} |b(x) - C_2|^{q'} dx \right)^{1/q'} |a(x)| dy \\ &\leq \int_Q \sum_{j=1}^{\infty} \frac{1}{2^j d} 2^j d^{n/q-n+1} \left(\frac{1}{2^j} + \int_{|x_0-y|/2^{j+1} d}^{|x_0-y|/2^j d} \frac{\omega_q(\delta)}{\delta} d\delta \right) |a(y)| dy \\ &\quad \times |Q_{j+1}^1|^{1/q'} \left(\frac{1}{|Q_{j+1}^1|} \int_{Q_{j+1}^1} |b(x) - C_2|^{q'} dx \right)^{1/q'} \\ &\leq C \left(1 + \int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta \right) \|b\|_{BMO} |Q|^{1/t'} \left(\int_Q |a(y)|^t dy \right)^{1/t} \leq C. \end{aligned}$$

On the other hand, by Hölder's inequality and the size condition of $a(x)$ again, it follows

$$\begin{aligned} I_{2,2} &\leq \int_Q \sum_{j=1}^{\infty} \frac{1}{2^j d} \left(\int_{2^j d \leq |x-x_0| \leq 2^{j+1} d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}} \right|^q dx \right)^{1/q} \times |Q_{j+1}^1|^{1/q'} |b(x) - C_2| |a(x)| dy \\ &\leq C \int_Q \sum_{j=1}^{\infty} \frac{1}{(2^j d)^{1-n/q'}} (2^j d)^{n/q-n+1} \left(\frac{1}{2^j} + \int_{|x_0-y|/2^{j+1} d}^{|x_0-y|/2^j d} \frac{\omega_q(\delta)}{\delta} d\delta \right) |b(x) - C_2| |a(x)| dy \\ &\leq C \left(\int_Q |b(x) - C_2|^{t'} dy \right)^{1/t'} \left(\int_Q |a(y)|^t dy \right)^{1/t} \leq C. \end{aligned}$$

Thus, we proved that $I \leq C$. Also, we complete the proof of Theorem 1.

REFERENCES

1. L. Hörmander, Estimates for translation invariant operators in L_p spaces, Acta. Math., 1960, 104: 93-140.
2. Y. Ding, S. Z. Lu and Q. Y. Xue, Boundedness of higher order commutators for Marcinkiewicz integral on Hardy spaces, Math., Res. Hot-Line, 2001, 5 (9), 47-58.
3. X. F. Shi and Y. S. Jiang, Weighted boundedness of parametric Marcinkiewicz integral and higher order commutators, Anal. Theory Appl., 2009, 1: 25-39.
4. E. M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc., 88 (1958), 430-466.
5. Y. Ding, D. Fan and Y. Pan, L_p boundedness of Marcinkiewicz integral with Hardy space function kernel, Acta. Math., Sinica, (English series), 16 (2000) 593-600.

6. D. Kurtz and R. Wheeden, Result on weight norm inequalities for multipliers, Trans. Amer. Math. Soc., 255 (1979), 343-362.
7. C. Pe´rez, Endpoint estimates for commutators of singular integral operators, Trans. Funct. Anal., 128 (1995), 163-185.
8. M. Paluszynski, Characterization of Lipschitz spaces via commutator of Coif-man, Rochbery, and Weiss: a Multiplier theorem for the semigroup of contractions, Thesis, Washington University, 1992.
9. J. Alvarez, Continuity properties for linear commutators of Caldero´n-Zygmund operators, Collect. Math., 49 (1998), 17-31.

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