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BOUNDEDNESS OF HIGHER ORDER COMMUTATORS FOR THE PARAMETRIC MARCINKIEWICZ INTEGRAL ON HARDY SPACE

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ABSTRACT

The authors study the commutators and higher order commutators of para-metric Marcinkiewicz integrals and obtain that the operators bounded from $H_b^1(R^n)$ to $L^p(R^n)$ with the symbol $b \in BMO$.

Keywords: *Hardy space, commutators, parametric Marcinkiewicz integral.*

1. INTRODUCTION

Let T be a Calderon-Zygmund singular integral operator. For a locally integral function b on R^n; the commutator [b; T] of T is defined by

$$[b,T]f(x) = b(x)T(f)(x) - T(bf)(x).$$

It is well known that the Calderon-Zygmund singular integral operator T is bounded from $H^1(R^n)$ to $L^1(R^n)$. However, it was observed in [7] that the corresponding result for [b, T] is false when b is a BMO function. In 1995, Perez in [7] introduced a subspace $H_b^1(R^n)$; and proved that [b, T] is bounded operator from $H_b^1(R^n)$ to $L^1(R^n)$ for $b \in BMO$, Following the definition of [8]. in 1998, Alvarez defined the atomic space $H_b^p(R^n)$, for $0 < p < 1$, and proved that the commutator $[b,T]$ is also bounded from $H_b^p(R^n)$ to $L^p(R^n)$, for $n/(n+1) < p \le 1$ in [9].

Before stating our results, let us first give the definitions and some known results on μ_ρ and $\mu_\rho^{b_m}$. Suppose that S^{n-1} is the unit sphere in $R^n (n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be a homogeneous function of degree zero on R^n satisfying $\Omega \in L^1(S^{n-1})$ and the following property

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0 \tag{1}$$

In 1960, H•omander in [1] introduced the parametric Marcinkiewicz integrals;

$$\mu_\rho(f)(x) = \left(\int_0^\infty |F_{\rho,t}(x)|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}}$$

where $0 \prec \rho \prec n$ and

$$F_{\rho,t}(f)(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy$$

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If $\rho=1$, then is the classical Marcinkiewicz integrals defined by Stein in [4]. Stein proved that if Ω is continuous and satisfies a $\mathrm{Lip}_a(0 \prec a \le 1)$ condition on S^{n-1}, then is of type (p,p) for 1 < p ≤ 2, and of weak type (1,1). On the other hand, let $b \in BMO(R^n)$ and $m \in N$, the m order commutators generated by the parametric Marcinkiewicz integrals μ_ρ and b is defined by

$$\mu_\rho^{b^m} f(x) = \left(\int_0^\infty |F_{\rho,t}^{b^m}(f)(x)|^2 \frac{dt}{t^{2\rho+1}} \right)^{\frac{1}{2}}$$

Where

$$F_{\rho,t}^{b^m}(f)(x) = \int_{|x-y|\le t} [b(x)-b(y)]^m \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy$$

In [2], Ding, Lu and Xue obtained boundedness of commutators for the Marcinkiewicz integrals on Hardy space. In [3], Shi and Jiang obtained the weighted L^p boundedness of μ_ρ and $\mu_\rho^{b^m}$ with rough kernels. In this paper, we will consider the boundedness of the higher order commutator $\mu_\rho^{b^m}$ on the atomic-Hardy spaces $H_{b^m}^1(R^n)$.

To state our results we need some notations and definitions. A function $\Omega(x')$ on S^{n-1} is said to satisfy the L^q-Dini condition $q \succ 1$, if $\Omega(x') \in L^q(S^{n-1})$ and $\int_0^1 \frac{\omega_q(\sigma)}{\sigma} d\sigma \prec \infty$ (2)

Where $\omega(\sigma) = \sup\limits_{|\rho|\le\sigma} (\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q \, d\sigma(x'))^{\frac{1}{q}}$

and is a rotation on $S^{n-1}, |\rho| = \|\rho - I\|$, *We say that* $\Omega \in Lip(S^{n-1})$ *for* $0 \prec \alpha \le 1$.
If there exists a constant M>0 such that for any

$$x, y \in S^{n-1}, |\Omega(x) - \Omega(y)| \le M|x-y|^\alpha.$$

Definition 1: Let $b \in BMO(R^n)$, $m \in N$ and $1 \le t \le \infty$. A function $a(x)$ is said to be a (1, t, b^m)-atoms, if it satisfies the following conditions:

$$(1) \sup p(a) \subset Q(x_0, r);$$

$$(2) \|a\|_t \le Q^{1/t-1};$$

$$(3) \int_{R^n} a(x)dx = \int_{R^n} a(x)b(x)dx = \cdots = \int_{R^n} a(x)b^m(x)dx = 0,$$

Where Q is a ball in R^n.

Now let us state our main results.

Theorem 1: Let Ω is a homogeneous function of degree zero on R^n satisfying (1) and the L^q-Dini condition for $q \succ 1$, Then there exists a constant $C \succ 0$, independent of f such that $\|\mu_\rho^{b^m}(f)\|_{L^1} \le C\|f\|_{H_{b^m}^1}$.

2. PROOF OF THEOREMS

Because the proof of the following Lemma is very similar to Lemma 5 in [6], we just formulate it.

Lemma 1: *Suppose that is homogeneous of degree zero and satisfies* L^q*-Dini condition for* $q \succ 1$, *If there exist a constant* α_0, $0 \prec \alpha_0 \prec 1/2$, *such that* $|x| \prec \alpha_0 R$, *then*

$$\left(\int_{R \prec |y| \prec 2R} \left| \frac{\Omega(x-y)}{|y-x|^{n-p}} - \frac{\Omega(y)}{|y|^{n-\rho}} \right|^q \mathrm{dy} \right)^{\frac{1}{q}} \le CR^{n/q-(n-1)} \left(\frac{|x|}{R} + \int_{|x|/2R \prec \delta \prec \pi|x|/R} \frac{\omega_q(\delta)}{\delta} d\delta \right),$$

where the constant $C \succ 0$ is independent of R and x.

When $\omega \equiv 1$, from [3], we know the following conclusion immediately.

Lemma 2: Suppose that Ω be a homogeneous function of degree zero on R^n satisfying (1) and $0<\rho<n$. If $\Omega \in L^q(S^{n-1})(q>1)$ and $m \in N, b(x) \in BMO(R^n)$.
Then for $1<p<\infty$, there exists a constant $C>0$, independent of f, such that $\|\mu_\rho^{b^m}(f)\|_p \le C\|f\|_p$.

Now, let us turn to the proof of Theorem 1 only for the case $m=1$, When $m>1$, the proof is similar to the case $m=1$, but more complex in the form and we omit the details by brevity.

Proof of Theorem 1: It suffices to prove that $\|\mu_\rho(a)(x)\|_{L^1} \le C$ for any $(1,t,b)$-atom $a(x)(1<t<\infty)$. Fix a ball $Q=Q(x_0,r) \subset R^n$, and let $Q^* = 8n^{\frac{1}{2}}Q$, we have

$$\int_{R^n} \mu_\rho^b(a)(x)dx = \int_{Q^*} \mu_\rho^b(a)(x)dx + \int_{(Q^*)^c} \mu_\rho^b(a)(x)dx.$$

By Lemma 2, $H\ddot{o}lder's$ inequality and size condition of $a(x)$, we get

$$\int_{Q^*} \mu_\rho^b(a)(x)dx \le |Q^*|^{1/t'} \left(\int_{Q^*} |\mu_\rho^b(a)(x)|^t dx\right)^{1/t}$$
$$\le |Q|^{1/t'} \left(\int_{R^n} |a(x)|^t dx\right)^{1/t} \le |Q|^{1/t'} |Q|^{1/t-1} \le C.$$

Where $\frac{1}{t}+\frac{1}{t'}=1$. It remains to show that

$$I := \int_{(Q^*)^c} \mu_\rho(a)(x)dx \le C.$$

Taking that $(a+b)^l \le a^l + b^l$, when $a,b \ge 0$ and $0<l\le 1$, we have

$$I \le \int_{(Q^*)^c} \left(\int_0^{|x-x_0|+2r} \left| \int_{|x-y|\le t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y)dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} dx$$
$$+ \int_{(Q^*)^c} \left(\int_{|x-x_0|+2r}^{\infty} \left| \int_{|x-y|\le t} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y)dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} dx$$

Since $|x-y| \sim |x-x_0| \sim |x-x_0|+2r$ for any $x \in Q^*$ and $y \in Q$, we have

$$\left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_0|+2r)^{2\rho}} \right| \le C\frac{r}{|x-y|^{2\rho+1}}. \tag{3}$$

By (3), we have

$$I_1 \le \int_{(Q^*)^c} \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |b(x)-b(y)||a(y)| \left(\int_{|x-y|\le t \le |x-x_0|+2r} \frac{dt}{t^{2\rho+1}} \right)^{1/2} dydx$$
$$\le \int_{(Q^*)^c} \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |b(x)-b(y)||a(y)| \left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_0|+2r)^{2\rho}} \right|^{1/2} dydx$$
$$\le Cd^{1/2} \int_Q \int_{(Q^*)^c} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} |b(x)-b(y)| dx |a(y)| dy$$

$$\le Cd^{1/2}\int_Q\sum_{j=1}^{\infty}\int_{2^j d\le|x-y|\le 2^{j+1}d}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}|b(x)-C_1|dx|a(y)|dy$$

$$+Cd^{1/2}\int_Q\sum_{j=1}^{\infty}\int_{2^j d\le|x-y|\le 2^{j+1}d}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}|b(y)-C_1|dx|a(y)|dy$$

$$=I_{1,1}+I_{1,2},$$

Where $C_1=\frac{1}{|Q_{j+1}|}\int_{Q_{j+1}}|b(x)|dx$ and $Q_{j+1}=\{x:|x-y|<2^{j+1}d\}$. Below we give the estimates of $I_{1,1}$ and $I_{1,2}$.

Using $H\ddot{o}lder's$ inequality and the size condition of $a(x)$, We get

$$I_{1,1}\le Cd^{1/2}\int_Q\sum_{j=1}^{\infty}\left(\int_{2^j d\le|x-y|\le 2^{j+1}d}\frac{|Q(x-y)|^q}{|x-y|^{n+1/2}}dx\right)^{1/q}\times\left(\frac{|Q_{j+1}|}{|Q_{j+1}|}\int_{2^j d\le|x-y|\le 2^{j+1}d}\frac{|b(x)-C_1|^{q'}}{|x-y|^{n+1/2}}dx\right)^{1/q'}|a(y)|dy$$

$$\le Cd^{1/2}\sum_{j=1}^{\infty}\frac{(2^{j+1}d)^{n/q}(2^{j+1}d)^{n/q'}}{(2^j d)^{(n+1/2)/q}(2^j d)^{(n+1/2)/q'}}\|b\|_{BMO}\int_Q|a(y)|dy$$

$$\le C\sum_{j=1}^{\infty}\frac{1}{(2^j)^{1/2}}\|b\|_{BMO}\int_Q|a(y)|dy$$

$$\le C\|b\|_{BMO}|Q|^{1/t'}\left(\int_Q|a(y)|^t dy\right)^{1/t}\le C.$$

On the other hand, note that

$$\int_{2^j d\le|x-y|\le 2^{j+1}d}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}dx\le C\left(\frac{1}{(2^j)^{1/2}}\right).$$

SO

$$I_{1,2}\le Cd^{1/2}\int_Q\left(\sum_{j=1}^{\infty}\frac{1}{(2^j)^{1/2}}\right)|b(y)-C_1||a(y)|dy$$

$$\le C\left(\int_Q|b(y)-C_1|^{t'}dy\right)^{1/t'}\left(\int_Q|a(y)|^t dy\right)^{1/t}$$

$$\le C.$$

For I_2, using the vanishing condition of a(x), we get

$$I_2=\int_{(Q^*)^c}\left(\int_{|x-x_0|+2r}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}a(y)dy-\int_{|x-y|\le t}\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}(b(x)-b(y))a(y)dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{\frac{1}{2}}dx$$

$$\le C\int_{(Q^*)^c}\int_{R^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|\times|b(x)-b(y)||a(y)|\left(\int_{|x-x_0|+2r}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{\frac{1}{2}}dydx$$

$$\le C\int_{(Q^*)^c}\int_{R^n}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|\frac{|b(x)-b(y)||a(y)|}{|x-x_0|}dydx$$

$$\le C\int_Q \sum_{j=1}^{\infty}\int_{2^j d\le|x-x_0|\le 2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|\left|b(x)-C_2\right|dx\left|a(y)\right|dy$$

$$+C\int_Q \sum_{j=1}^{\infty}\int_{2^j d\le|x-x_0|\le 2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|\left|b(x)-C_2\right|dx\left|a(y)\right|dy$$

$$=I_{2,1}+I_{2,2}\,.$$

Where $C_3=\dfrac{1}{Q_{j+1}^1}\int_{Q_{j+1}^1}\left|b(x)\right|dx$ and $Q_{j+1}^1=\left\{x:\left|x-x_0\right|<2^{j+1}d\right\}$.By $H\ddot{o}lder's$

Inequality, Lemma 1 and the size condition of $a(x)$, we have

$$I_{2,1}\le C\int_Q \sum_{j=1}^{\infty}\frac{1}{2^j d}\left(\int_{2^j d\le|x-x_0|\le 2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|^q dx\right)^{1/q}$$

$$\times\left(\int_{2^j d\le|x-x_0|\le 2^{j+1}d}\left|b(x)-C_2\right|^{q'}dx\right)^{1/q'}\left|a(x)\right|dy$$

$$\le\int_Q \sum_{j=1}^{\infty}\frac{1}{2^j d}2^j d^{\,n/q-n+1}\left(\frac{1}{2^j}+\int_{|x_0-y|/2^{j+1}d}^{|x_0-y|/2^j d}\frac{\omega_q(\delta)}{\delta}d\delta\right)\left|a(y)\right|dy$$

$$\times\left|Q_{j+1}^1\right|^{1/q'}\left(\frac{1}{\left|Q_{j+1}^1\right|}\int_{Q_{j+1}^1}\left|b(x)-C_2\right|^{q'}dx\right)^{1/q'}$$

$$\le C\left(1+\int_0^1\frac{\omega_q(\delta)}{\delta}d\delta\right)\|b\|_{BMO}\left|Q\right|^{1/t'}\left(\int_Q\left|a(y)\right|^t dy\right)^{1/t}\le C\,.$$

On the other hand, by $H\ddot{o}lder's$ inequality and the size condition of $a(x)$ again, it follows

$$I_{2,2}\le\int_Q \sum_{j=1}^{\infty}\frac{1}{2^j d}\left(\int_{2^j d\le|x-x_0|\le 2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|^q dx\right)^{1/q}\times\left|Q_{j+1}^1\right|^{1/q'}\left|b(x)-C_2\right|\left|a(x)\right|dy$$

$$\le C\int_Q \sum_{j=1}^{\infty}\frac{1}{\left(2^j d\right)^{1-n/q'}}\left(2^j d\right)^{n/q-n+1}\left(\frac{1}{2^j}+\int_{|x_0-y|/2^{j+1}d}^{|x_0-y|/2^j d}\frac{\omega_q(\delta)}{\delta}d\delta\right)\left|b(x)-C_2\right|\left|a(x)\right|dy$$

$$\le C\left(\int_Q\left|b(x)-C_2\right|^{t'}dy\right)^{1/t'}\left(\int_Q\left|a(y)\right|^t dy\right)^{1/t}\le C\,.$$

Thus, we proved that $I\le C$. Also, we comple the proof of Theorem 1.

REFERENCES

1. L. Ho¨mander, Estimates for translation invariant operators in Lp spaces, Acta.Math., 1960, 104: 93-140.
2. Y. Ding, S. Z. Lu and Q. Y. Xue, Boundedness ofhigher order commutators for Marcinkiewicz integral on Hardy spaces, Math., Res. Hot-Line, 2001, 5 (9), 47-58.
3. X. F. Shi and Y. S. Jiang, Weighted boundedness ofparametric Marcinkiewicz integral and higher order commutators, Anal. Theory Appl., 2009, 1: 25-39.
4. E. M. Stein, On the function ofLittlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc., 88 (1958), 430-466.
5. Y. Ding, D. Fan and Y. Pan, Lp boundedness ofMarcinkiewicz integral with Hardy space function kernel, Acta. Math., Sinica, (English series), 16 (2000) 593-600.

6. D. Kurtz and R. Wheeden, Result on weight norm inequalities for multipliers, Trans. Amer. Math. Soc., 255 (1979), 343-362.
7. C. Pe´rez, Endpoint estimates for commutators ofsingular integral operators, Trans. Funct. Anal., 128 (1995), 163-185.
8. M. Paluszynski, Characterization ofLipschitz spaces via commutator of Coif-man, Rochbery, and Weiss: a Multiplier theorem for the semigroup ofcontrac- tions, Thesis, Washington University, 1992.
9. J. Alvarez, Continuity properties for lingar commutators of Caldero´n-Zygmund operators, Collect. Math., 49 (1998), 17-31.

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