Research Journal of Pure Algebra -1(8), 2011, Page: 171-175
Available online through www.rjpa.info ISSN 2248–9037

FINITE GROUPS WITH A MAXIMAL NILPOTENT SUBGROUP

Runshi Zhang, Deqin Chen and Shitian Liu*

School of Science, Sichuan University of Science & Engineering, Zigong, 643000, China

E-mails: zhangrs-75@suse.edu.cn, chendeqin@suse.edu.cn; liust@suse.edu.cn

(Received on: 22-10-11; Accepted on: 07-11-11)

ABSTRACT

A solvability condition for finite groups with maximal nilpotent subgroups is given as follows. Let G be a finite group and M a maximal nilpotent subgroup of G. If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is pronormal in G, then G is solvable.

Keywords: *maximal nilpotent subgroups, solvable groups, pronormal subgroups*

Subject Classification (2010): *20D10, 20D20.*

1. INTRODUCTION:

All groups considered are finite. Hall in [7] introduced the concept of pronormality. Criteria for pronormality were given and studied by some authors (see [12, 13] and [10, 17]). D'Anielio introduced (see[5]) the concept of dual pronormality, and gave the structure of finite groups such that the n-maximal subgroups are dual pronormal. D'Anielio introduced (see[5]) the concept of $\mathbb{F}$-dual pronormality, and gave the structure of finite groups such that the n-maximal subgroup is $\mathbb{F}$-dual pronormal. Bianchi etc. in [3] introduced the concept of H-subgroup. H-subgroups were studied by Asaad in [2], Csrogo and Herzog in[4], and Guo and Wei in [9].

A group is said to be solvable if its composition factors are all of prime order (see [8]). Let G be a finite group all of whose proper subgroups are nilpotent. Then G is soluble (see [16, 9.19]). Thompson in [18] proved that ifi G is a finite group with a maximal nilpotent subgroup of odd order, then G is soluble. Deskins in [6] proved that G has a maximal nilpotent subgroup of class ≤ 2, then G is soluble. Asaad in [1] proved that if G is a finite group of odd order n in which every minimal subgroup is pronormal in G, then G is supersolvable. A nature question arises:

If $2 \| G |$, certain subgroups are pronormal in G, what can be said about the structure of G ?

The key of this note is to prove the following:

Main Theorem: Let G be a finite group and M a maximal nilpotent subgroup of G. If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is pronormal in G, then G is solvable.

***Corresponding author:** Shitian Liu*,*E-mail:** liust@suse.edu.cn

2. PRELIMINARIES:

The concept of pronormality is one of the most important of embedding properties and was first introduced by P. Hall in his lectures in Cambridge.

Definition 2.1[7, I, 6.1] Let G be a group and $U \le G$. Then U is said to be pronormal in G (written U pr G) if, for each $g \in G$, the subgroups U and U^g are conjugate in their join $\langle U, U^g \rangle$.

Remark2.1 Normality implies pronormality. But the converse is not the case. For example, let $G = A_4$, the alternate group of degree 4. Obviously, the Sylow 2-subgroup P of G are pronormal in G (see[8,p13]), but not normal in G. If P is normal in G, then G must have element of order 6, which is impossible.

Lemma 2.1 [7, I, 6.3] Let U be a pronormal subgroup of a group G.

(1) If $U \le L \le G$, then U pr L;

(2) If $U \le K \lhd G$, then $G = N_G(U)K$; in other words, the Frattini argument applies to pronormal subgroups;

(3) If $K \lhd G$, then UK pr G; furthermore, UK/K pr G/K and $N_G(UK) = N_G(U)K$;

(4) If U is subnormal in G, then $U \lhd G$;

(5) $N_G(U)$ is both pronormal and self-normalizing in G.

Let rad (G) denote the radical of a group G, which is the largest solvable normal subgroup of G.

Lemma 2.2 [15, Theorem] If a finite group G contains a maximal nilpotent subgroup M and a non-abelian minimal normal subgroup N and if P is the Sylow 2-subgroup of M, then $G = P$.

3. THE PROOF OF MAIN THEOREM AND ITS APPLICATIONS:

In this section, we will give the proof of the Main Theorem, some applications, and also any remarks.

The Proof of Main Theorem:

Proof: Assume that the result is false, and let G be a counterexample of minimal order. Then we have:

Step: 1 $O_{P'}(G) = 1$

If $O_{P'}(G) = 1$, Let Q be normal q-subgroup of $O_{P'}(G)$, where $q \ne p$, and so Q is normal in G since Q is characteristic in $O_{P'}(G)$. Then we consider the quotient group G/Q. Let P_2 be a subgroup of P of order 2 or 4 is pronormal in G. Then, by Lemma 2.1(c), every subgroup P_2Q/Q of PQ/Q of order 2 or 4 is pronormal in G/Q, where PQ/Q is a Sylow 2-subgroup of MQ/Q.

Then G/Q, PQ/Q satisfies the hypotheses of the theorem, and the minimal choice of G implies that G/Q is solvable. This means that G is solvable by [8, 4.1(ii) (iii), p23], a contradiction. Then $O_{P'}(G) = 1$.

Step: 2 For any maximal nilpotent subgroups M , $M/F(G)$ is solvable, where $F(G)$ is the Fitting subgroup of G . Furthermore, $\Phi(G)=F(G)$, where $\Phi(G)$ is the Frattini subgroup of G .

If $F(G)\le M$, then $G=MF(G)$ and M are solvable since the minimal choice of G . Thus G is solvable by [8, 4.1(ii)(iii), p23], a contradiction. Then, by Step 1, $M/F(G)$ is solvable. On the other hand, if $\Phi(G)>F(G)$ then there exists a normal subgroup W such that $W\le F(G)$ but W is not in $\Phi(G)$. And so we have G/W is soluble. And since $G/\Phi(G)$ is soluble, G is isomorphic to a subgroup of $G/\Phi(G)\times G/W$. Thus, by [16, 5.1.2], G is soluble, a contradiction. Then we have $F(G)=\Phi(G)$.

Step: 3 Let $R=rad(G)$, then $G=PR$, and P is a Sylow 2-subgroup of G .

By Lemma 2.2, we have $G=PR$, If P is not a Sylow 2-subgroup, then by the maximality of M , then $|G:M|=2^{\alpha}$ for some integer α .Then there exists a subgroup M_0 such that $M\le M_0\le G$ and $|M:M_0|=2$. Since for every subgroup L of P of order 2 or 4 is pronormal in G, then L is pronormal in M_0 by Lemma 2.1(a), the minimal choice of G implies that M_0 is soluble. By hypotheses, M is nilpotent, and so, since $M\lhd M_0$, M_0 is nilpotent, which contradicts the maximality of M . Thus P is a Sylow 2-subgroup of G.

Step: 4 the final contradiction.

By Step 2, $R\ge\Phi(G)$, and, by [11, Lemma 2.6], $F(R)$ is the direct product of minimal normal subgroups of G which is contained in R . Let $R=\langle x_1,x_2,\cdots x_n\rangle$, where $x_i\in R$ is of order $p_i\ne 2$ (p_i is prime number of G) by Step 2. Then $F(R)=\langle x_1\rangle\times\cdots\langle x_n\rangle$, and $\langle x_i\rangle$ char $F(R)$ char R . Thus $\langle x_i\rangle\lhd G$. Then we have $G/\langle x_i\rangle$ is soluble by Step 1, and so G is soluble, a contradiction.

This completes the proof.

Remark: 3.1 The condition of the theorem ‘nilpotent’ can't be removed. Let $G=A_5$, the alternate group of degree 5. The subgroup of the Sylow 2-subgroup of order 4 is pronormal in G, but G is not soluble.

Remark: 3.2 The condition of the theorem “nilpotent”can't be replaced by “soluble”. Let $G=A_5$, the alternate group of degree 5. Obviously, A_4 , the alternate group of degree 4, is the soluble maximal subgroup of G . And the subgroup of the Sylow 2-subgroup of order 4 is pronormal in G , but G is not soluble.

Corollary: 3.1 Let G be a finite group and M a maximal nilpotent subgroup of G . If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is normal in G , then G is solvable.

Proof: By Remark 2.1, normality must be pronormal. Then we use Main Theorem to Corollary 3.1, G is soluble. This completes the proof.

Corollary: 3.4 [14, Theorem] Let G be a finite group with a nilpotent maximal subgroup M. If P is a Sylow 2-subgroup of M and $G_2 \cap \mathrm{P}$ is normal in P for any Sylow 2-subgroup G_2 of G. Then G is soluble.

ACKNOWLEDGEMENT:

The object is supported by the Scientific Research Fund of School of Science of SUSE (Grant Number: 09LXYB02) and NSF of SUSE (Grant Number: 2010XJKYL017). The author is very grateful for the helpful suggestions of the referee.

REFERENCES:

[1]M. Asaad, On the supersolvability of finite groups, I, Acta Mathematica Academiae Scientiarum Hungaricae, 38(1-4) (1981), 57-59.

[2] M. Asaad, Some results on p-nilpotence and supersolvability of fintie groups, Commucations in Algebra, 34(2006), 4217-4224.

[3] M. Bianchi, A.G.B. Mauri, M. Herzog,and L. Verard, On finite solvable groups in which normality is a transitive relation, Journal of Group Theory, 3(2000), 147-156.

[4] P. Csorgo,and M. Herzog, On supersolvable groups and the nilpotator, Communications in Algebra, 32(2)(2004), 609-620.

[5] A. D'Anillo, and M.D. Perez-Ramos, On dual pronormal subgroups and Fitting classes, in "Groups St Andrews 2001 in Oxford, London Mathematical society Lecture Note Series 304", 1(2003), 92-100.

[6] W. E. Deskins, A condition for solvability of afinite group, Illinois Journal of Mathematics, 5(1961), 306-311.

[7] K. Doerk, and T. Hawkes, Finite soluble groups, Walter de Gruyter, New York, 1992.

[8] D. Gorenstein, Finite groups 2nd, Chelsea Publishing Corporation, Rhode Island, 1980.

[9] X. Guo, and X. Wei, The influence of H-subgroups on the structure of finite groups, Journal of Group Theory, 13(2) (2010), 267-276.

[10] L.A. Kurdachenko, and I.Y. Subbotin, On transitivity of pronormality, Commentationes Mathematicae Universitatis Carolinae, 43(4) (2002), 589-594.

[11] D. Li, and X. Guo, The influence of c-normality of subgroups on the structure of finite groups , Journal of Pure and Applied Algebra,150(2000), 53-60.

[12] A. Mann, A criterion for pronormality, Journal of the London Mathematical Society Ser 1, 44 (1969), 175-176.

[13] T. A. Peng, Finite groups with pro-normal subgroups, Proceedings of the American Mathematical Society, 20(1969), 232-234.

[14] J. Randolph, A solvability condition for finite groupswith nilpotent maximal subgroups, Bulletin of the Australian Mathematical Society, 3(1970), 273-276.

[15] J. Randolph, On a theorem of Thompson concerning a class of non-solvable groups, Mathematische Zeitschrift, 118(1970), 340-343.

[16] D. J. S. Robinson, A course in the theory of groups 2nd, Springer-Verlag, New York, 2004.

[17] M. I. Saluk, Finite groups with nonabnormal pairs of subgroups, Mathematical Notes, 31(1982), 179-182.

[18] J. G. Thompson, Normal p-complements for finite groups, Mathematische Zeitschrift, 72(1960), 332-354.

[19] G. J. Wood, On pronormal subgroups of finite soluble groups, Archiv der Mathematik, 25(1974), 578-585.

****

