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LEFT I-ORDERS IN STRONG SEMILATTICES OF PROPER BISIMPLE INVERSE SEMIGROUPS

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ABSTRACT

Let Q be an inverse semigroup. A subsemigroup S of Q is a left I-order in Q and Q is a semigroup of left I-quotients of S if every element $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in S$. We characterize semigroups Q which are left I-quotients of semigroups S which are strong semilattices of right cancellative monoids with the (LC) condition and certain further conditions. We give necessary and sufficient conditions for Q to be proper.

***Keywords**: I-orders, I-quotients, right cancellative monoid, inverse hull.*

INTRODUCTION

It is well known that a semigroup S has a group of left quotients if and only if S is cancellative and right reversible [1, Theorem 1.24]. By saying that a semigroup S is *right reversible* we mean for any $a, b \in S$, $Sa \cap Sb \neq \emptyset$. Inspired by methods of both classical ring and semigroup theory, Fountain and Petrich in [4] extended the notion of group of left quotients of a semigroup S to that of semigroup of left quotients of S. Their main idea is that we consider inverse of elements in any subgroup of a semigroup and not just the group of units (which may not exists). The definition of semigroups of left quotients proposed in [4] was restricted to completely 0-simple semigroups of left quotients. This was generalised to much wider class of semigroups by Gould [9]. The idea is that a subsemigroup S of a semigroup Q is a *left order* in Q if every element in Q can be written as $a^{\natural}b$ where $a, b \in S$ and $a^{\natural}$ is an inverse of a in a *subgroup* of Q and if, in addition, every square-cancellable element of S (an element a of a semigroup S is square-cancellable if $a\,\mathcal{H}^*a^2$) lies in a subgroup of Q. In this case we say that Q is a semigroup of *left quotients* of S. *Right orders* and *semigroup of right quotients* are defined dually. If S is both a left and right order in Q, then S is an *order* in Q and Q is a *semigroup of quotients* of S.

The author and Gould in [5] have introduced the following definition of left I-orders in inverse semigroups: A subsemigroup S of an inverse semigroup Q is a *left I-order* in Q and Q is a semigroup of *left I-quotients* of S if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of an inverse semigroup theory. *Right I-orders* and semigroups of *right I-quotients* are defined dually. If S is a left and right I-order in an inverse semigroup Q, we say that S is an *I-order* in Q and Q is a semigroup of *I-quotients* of S. Let S be a left I-order in Q. Then S is *straight* in Q if every $q \in Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a\,\mathcal{R}\,b$ in Q.

This definition has been used to describe left I-orders in various classes of inverse semigroups, for example in [2] and [6].

Clifford [1] showed that any right cancellative monoid S with the (LC) condition is the $\mathcal{R}$-class of the identity of its inverse hull $\sum(S)$. Moreover, (in our terminology) S is a left I-order in $\sum(S)$. By saying that a semigroup S has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. Clifford established that precisely bisimple inverse monoids can be regarded as inverse hulls of right cancellative monoids S satisfying the (LC) condition. The author and Gould in [7] have extended Clifford's work to a left ample semigroup with (LC). It is worth pointing out that the inverse hull of the left ample semigroup need not be bisimple

It was shown in [1] that a semigroup Q which is a semilattice $Y$of inverse semigroups Q_α is an inverse semigroup, but if each Q_α is proper Q may not be proper (see, Example 5.2 [15]).

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Let Q be a strong semilattices Y of bisimple inverse monoids Q_α, such that the set of identities elements forms a subsemigroup. Let S be a strong semilattices Y of right cancellative monoids $S_\alpha, \alpha \in Y$ with (LC) condition and certain morphisms satisfying two conditions. In [10] Gantos showed how to recover the structure of Q from that of S; in our terminology, Q is a semigroup of left I-quotients of S. The purpose of this paper is to study the case where $Q_\alpha, \alpha \in Y$ is proper. We give the conditions which make Q is proper, by using the structure of S.

The rest of this article is structured as follows. Section 2 contains preliminaries Section 3 contains the main results of the paper.

2. PRELIMINARIES

We begin by recalling some of the basic facts about the relations $\mathcal{R}^*$ and $\mathcal{L}^*$. Let S be a semigroup and $a, b \in S$. We call elements a and b to be related by $\mathcal{R}^*$ if and only if a and b are related by $\mathcal{R}$ in some oversemigroup of S. Dually, we can define the relation $\mathcal{L}^*$. An alternative description of $\mathcal{R}^*$ is provided by the following lemma.

Lemma 2.1 [3] Let S be a semigroup and $a, b \in S$. Then the following are equivalent

(i) $a\, \mathcal{R}^* b$;

(ii) for all $x, y \in S^1$ $xa = ya$ if and only if $xb = yb$.

It is well-known that Green star relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on a semigroup S are generalizations of the usual Green's relations $\mathcal{R}$ and $\mathcal{L}$ on S, respectively.

A semigroup S is *left adequate* if every $\mathcal{R}^*$-class of S contains an idempotent and the idempotents $E(S)$ of S form a semilattice. In this case every $\mathcal{R}^*$-class of S contains a unique idempotent. We denote the idempotent in the $\mathcal{R}^*$-class of a by a^+. A left adequate monoid S is *left ample* if $(ae)^+a = ae$ for each $a \in S$ and $e \in E(S)$.

We can note easily that, any right cancellative monoid is left ample. By a right cancellative semigroup we mean, a semigroup S such that for all $x, y \in S$

$$xz = yz \text{ implies } x = y.$$

Following [8], for any left ample semigroup S we can construct an embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$ we let $\rho_a \in \mathcal{I}_S$ be given by

$$\text{dom}\ \rho_a = Sa^+ \ and \ \text{im}\ \rho_a = Sa$$

and for any $x \in \text{dom}\ \rho_a$.

$$x\rho_a = xa.$$

Then the map $\theta_S: S \to \mathcal{I}_S$ is a (2,1)-embedding.

The inverse hull of a left ample semigroup S is the inverse subsemigroup $\sum(S)$ of $\mathcal{I}_S$ generated by im θ_S. If S is a right cancellative monoid, then for any $a \in S$ we have $a^+ = 1$. Then $\rho_a: S \to Sa$ is defined by

$$x\rho_a = xa \text{ for each } x \text{ in } S.$$

Hence dom $\rho_a = S =$ dom I_S, giving that im $\theta_S \subseteq R_1$ where R_1 is the $\mathcal{R}$ -class of I_S in $\mathcal{I}_S$.

As in [5] we say that a (2, 1)-morphism $\phi: S \to T$, where S and T are left ample semigroups with Condition (LC), is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap S(c\phi) = S(w\phi).$$

Let S be a left I-order in an inverse semigroup Q. To emphasis that $\mathcal{R}$ and L are relations Q, we may write $\mathcal{R}^Q$ and $\mathcal{L}^Q$ or $\mathcal{R}$ in Q and $\mathcal{L}$ in Q.

Let $\sum(S)$ be the inverse hull of left I-quotents of a right cancellative monoid S with (LC). In the rest of this article_we identify S with $S\theta_S$, where θ_S is the embedding of S into I_S. We write $a^{-1}b$ short for the element $\rho_a{}^{-1}\rho_b$of $\sum(S)$ where $a, b \in S$.

An inverse semigroup S is called *proper* if for any a in S and e in $E(S)$, $ae\epsilon E$ implies that $a\epsilon E$ Munn [14] showed that the relation

$$\sigma = \{(a, b) \in S \times S: ea = eb \text{ for some } e^2 = e \in S\}$$

is the *minimum group congruence* on any inverse semigroup S, that is, σ is the smallest congruence on S such that $S/_\sigma$ is a group.

We now give some an alternative condition for an inverse semigroup to be proper.

Proposition 2.2: [13] The following are equivalent for an inverse semigroup S:
(1) S is proper;
(2) $\sigma \cap \mathcal{R} = I_S$, where I_S is the identity relation on S.

Let Q be an inverse monoid with identity 1, and let R_1 be the $\mathcal{R}$-class of the identity. Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R_1$. Since $a, b, c, d \in R_1$we have that

$$a^{-1} \mathcal{R}\, a^{-1}b = c^{-1}d \,\mathcal{R}\, c^{-1} \; in\; Q.$$

Then $a \,\mathcal{L}\, c$ in Q. Since $a \,\mathcal{R}\, b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that $ac^{-1}$is a unit. As $a \,\mathcal{L}\, c$, it follows that $ac^{-1}\mathcal{L}\, cc^{-1} = 1$. Since $c^{-1} \,\mathcal{R}\, c^{-1}$ we have that $1 = ac^{-1} \,\mathcal{R}\, ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a \,\mathcal{L}\, c$ we have that $uc = ac^{-1}c = a$. The converse is clear.

Lemma 2.3: [7] Let Q be an inverse monoid. Let $a, b, c, d \in R_1$. Then $a^{-1}b = c^{-1}d \; if\; and\; only\; if\; a = uc \; and$ $b = ud,$ for some unit u.

Lemma 2.4: Let S be a left I-order in Q. Let $q = a^{-1}b$ in Q where $a, b \in S$. Then $a\, \mathcal{R}^Q b$ if and only if $b\, \mathcal{L}^Q\, q\, \mathcal{R}^Q a^{-1}$. Consequently, S intersects every $\mathcal{L}$-class of Q.

The author and Gould have showed that a left ample semigroup with (LC) condition is a left I-orders in its inverse hull (see Theorem 3.7 of [5]). They extended the result of the following lemma.

Lemma 2.5: [5] The following conditions are equivalent for a right cancellative monoid S:
(*i*) $\sum(S)$ is bisimple;
(*ii*) S has Condition (LC);
(*iii*) S is a left I-order in $\sum(S)$.
If the above conditions hold, then S is the $\mathcal{R}$-class of Further, $\sum(S)$ is proper if and only if S is cancellative.

Further, $\sum(S)$ is proper if and only if S is cancellative.

Conversely, the $\mathcal{R}$-class of the identity of any bisimple inverse monoid is right cancellative with Condition (LC).

Theorem 2.6: [5] Let $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$ be a strong semilattice of left ample semigroups S_α, such that the connecting morphisms are (2, 1)-morphisms. Suppose that each S_α, $\alpha \in Y$ has (LC) and that S has (LC). For each $\alpha \in Y$, let $\sum_\alpha$ be the inverse hull of S_α. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, we have that $\varphi_{\alpha,\beta}$ lifts to a morphism $\phi_{\alpha,\beta}: \sum_\alpha \to \sum_\beta$. Further, $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ is a strong semilattice of inverse semigroups, such that S is a straight left I-order in Q. Moreover, Q is isomorphic to the inverse hull of S.

Since right cancellative monoids are precisely left sample semigroups possessing a single idempotent. The following corollary is clear.

Corollary 2.7: Let $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$ and for each α, let S_α be a right cancellative monoid with Condition (LC) and $\sum_\alpha$ as its inverse hull of left I-quotients. Suppose that S has the (LC) condition. Then S is a straight left I-order in a strong semilattice of monoids $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ where $\varphi_{\alpha,\beta}$'s lift to $\phi_{\alpha,\beta}$'s, $\alpha \geq \beta$.

3. SEMILATTICES OF PROPER BISIMPLE INVERSE SEMIGROUPS

Let $S = \left[Y; S_\alpha; \varphi_{\alpha,\beta}\right]$ be a strong semilattice Y of right cancellative monoids S_α, $\alpha \in Y$ with the (LC) condition and S has (LC). From Corollary 2.7, S has a strong semilattice of left I-quotients $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ where $\sum_\alpha$ is the inverse hull of S_α for each $\alpha \in Y$ and each $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$. We recall that the connecting morphism $\varphi_{\alpha,\beta}$ is given by $a\varphi_{\alpha,\beta} = e_\beta a$. We employ this section to study the case when Q is proper.

Theorem 3.1: Let $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$, where each S_α is a right cancellative monoid with Condition (LC) and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let $\sum_\alpha$ be the inverse hull of left I-quotients of S_α for each $\alpha \in Y$. Then $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ is a semigroup of left I-quotients of S. Moreover, each $\phi_{\alpha,\beta}$ is one-to-one and each $\sum_\alpha$ proper if and only if Q is proper and $(\star)$ holds where $(\star)$ is the following condition: for all $a, b \in S_\alpha$ and for all $\alpha \geq \beta$, $a\phi_{\alpha,\beta}\, \mathcal{L}^{\Sigma_\beta}\, b\phi_{\alpha,\beta}$ *implies that* $a\, \mathcal{L}^{\Sigma_\alpha} b$.

Proof: By Lemma 2.5, each S_α is a left I-order in its inverse hull $\sum_\alpha$ and S_α is the $\mathcal{R}$-class of the identity of $\sum_\alpha$. From Corollary 2.7, we have that S is a left I-order in Q. Suppose that Q is proper and $(\star)$ holds. To show that each $\phi_{\alpha,\beta}$ is one-to-one, let

$$(a^{-1}b)\phi_{\alpha,\beta} = (c^{-1}d)\phi_{\alpha,\beta}$$

where $a^{-1}b, c^{-1}d \in \sum_\alpha$ for some $a, b, c, d \in S_\alpha$. Since $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y, we have

$$(e_\beta a)^{-1}(e_\beta b) = (e_\beta c)^{-1}(e_\beta d).$$

Hence $a^{-1}e_\beta b = c^{-1}e_\beta d$ and so $a^{-1}be_\beta = c^{-1}de_\beta$ as the identities are central in Q. It follows that $a^{-1}b \, \sigma \, c^{-1}d$ in Q. Using Lemma 2.4, we get $a\phi_{\alpha,\beta} = e_\beta a \, \mathcal{L}^{\Sigma_\beta} \, e_\beta c = c\phi_{\alpha,\beta}$, by assumption. Since $\mathcal{R}^{\Sigma_\alpha} = \mathcal{R}^Q \cap (\sum_\alpha \times \sum_\alpha)$ and $\mathcal{L}^{\Sigma_\alpha} = \mathcal{L}^Q \cap (\sum_\alpha \times \sum_\alpha)$ for each $\alpha \in Y$ (see, Proposition 2.4.2 of [11]) we have that $a \, \mathcal{L}^Q \, c$ and so $a^{-1}\mathcal{R}^Q c^{-1}$. Again by Lemma 2.4,

$$a^{-1}b \, \mathcal{R}^Q \, a^{-1}\mathcal{R}^Q c^{-1}\mathcal{R}^Q c^{-1}d,$$

so that $a^{-1}b \, \mathcal{R}^Q c^{-1}d$. Since Q is proper and $a^{-1}b \, (\sigma \cap \mathcal{R}^Q) \, c^{-1}d$, it follows that $a^{-1}b = c^{-1}d$, by Proposition 2.2.

Hence $\phi_{\alpha,\beta}$ is one-to-one. It is clear that if Q is proper, then $\sum_\alpha$ is proper for all $\alpha \in Y$.

On the other hand, suppose that each $\phi_{\alpha,\beta}$ is one-to-one and $\sum_\alpha$ is proper for each $\alpha \in Y$. To show that Q is proper, let

$$a^{-1}bc^{-1}c = c^{-1}c$$

where $a^{-1}b \in \sum_\alpha$ and $c^{-1}c \in \sum_\beta$ for some $a, b \in S_\alpha$ and $c \in S_\beta$. It is clear that $\beta \leq \alpha$ and so $S_{\alpha\beta} = S_\beta$. By definition of multiplication

$$a^{-1}bc^{-1}c = (xa)^{-1}(yc) = c^{-1}c$$

where $xb = yc$ for some $x, y \in S_{\alpha\beta}$ and as $xa, yc, c \in S_\beta$ we have that $xa = uc = yc$ for some unit u in S_β, by Lemma 2.3. Since $xb = yc$ we have that $xe_\beta b = yc = xe_\beta a$, as $\sum_\beta$ is proper, it follows that S_β is cancellative, by Lemma 2.5. Hence $e_\beta b = e_\beta a$ and so $a\phi_{\alpha,\beta} = c\phi_{\alpha,\beta}$. Hence $a = b$ as $\phi_{\alpha,\beta}$ is one-to-one. To show that $(\star)$ holds, let $a\phi_{\alpha,\beta}\mathcal{L}^{\Sigma_\beta} \, c\phi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y and for all $a, b \in S_\alpha$. We have

$$(a^{-1}a)\phi_{\alpha,\beta} = (b^{-1}b)\phi_{\alpha,\beta}$$

So that as $\phi_{\alpha,\beta}$ is one-to-one $a^{-1}a = b^{-1}b$ in $\sum_\alpha$ and so $a \, \mathcal{L}^{\Sigma_\alpha} \, b$ as required.

Following [11], let P be a semilattice Y of inverse semigroups P_α, and let $\sigma_\alpha = \sigma(P_\alpha)$ be the minimum group congruence on P_α. Define τ on P by

$$a \, \tau \, b \iff p \, \sigma_\alpha \, q \text{ in } P_\alpha \text{ for some } \alpha \in Y.$$

It is shown in [11] that τ is a congruence on P and P/τ is a semilattice Y of groups P_α/σ_α. That is, $P/\tau = \bigcup_{\alpha \in Y}(P_\alpha/\sigma_\alpha)$. For any $a\sigma_\alpha \in P_\alpha/\sigma_\alpha$ and $b\sigma_\beta \in P_\beta/\sigma_\beta$. we have

$$(a\sigma_\alpha)(b\sigma_\beta) = (a\tau)(b\tau) = (ab)\tau = (ab)\sigma_{\alpha\beta}$$

Lemma 3.2: [11] Let P be a semilattice Y of proper inverse semigroups P_α, $\alpha \in Y$ and let τ be defined as above, so that P/τ is a semilattice Y of groups $G_\alpha = P_\alpha/\tau_\alpha$ and define the mappings $\psi_{\alpha,\beta}: G_\alpha \to G_\beta$ by $a \, \psi_{\alpha,\beta} = ae_\beta$ where $a \in G_\alpha$ and e_β denotes the identity of G_β. Then the following are equivalent:

(1) P is proper;
(2) P/τ is proper;
(3) $\psi_{\alpha,\beta}$ is one-to-one where $\alpha \geq \beta$.

Corollary 3.3: Let $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$, where each S_α is a right cancellative monoid with Condition (LC) and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ be the semigroup of left I-quotients of S where $\sum_\alpha$ be the inverse hull of S_α for each $\alpha \in Y$ and σ_α be defined as above for each $\alpha \in Y$. Then the following are equivalent:

(1) Each S_α is left cancellative and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$;
(2) Q is proper and $(\star)$ holds;
(3) Each $\sum_\alpha$ is proper and $\psi_{\alpha,\beta}: G_\alpha \to G_\beta$ is one-to-one for all $\alpha, \beta \in Y$ where $G_\alpha = \sum_\alpha/\sigma_\alpha$ for all $\alpha \in Y$ and $(\star)$ holds.

Proof:
$(\mathbf{1}) \Rightarrow (\mathbf{2})$. By Lemma 2.5, $\sum_\alpha$ is proper for each $\alpha \in Y$. Then (2) follows by Theorem 3.1.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. It is clear that if Q is proper, then $\sum_\alpha$ is proper for all $\alpha \in Y$. From Lemma 3.2, we have that $\psi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ where $\alpha \geq \beta$. Hence (3) holds.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$. By Lemma 2.5, each S_α is left cancellative. It remains to show that each $\phi_{\alpha,\beta}$ is one-to-one. Since $\psi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ where $\alpha \geq \beta$ we have that Q is proper, by Lemma 3.2. Hence (1) holds by Theorem 3.1.

The following Lemma can be considered as a partial generalization of Lemma 2.5.

Lemma 3.4: Let $S=[Y;S_\alpha;\varphi_{\alpha,\beta}]$, where each S_α is a right cancellative monoid with Condition (LC) and each $\varphi_{\alpha,\beta}$ is (LC)-preserving. Let $Q=[Y;\sum_\alpha;\phi_{\alpha,\beta}]$ be the semigroup of left I-quotients of S where $\sum_\alpha$ be the inverse hull of S_α for each $\alpha\in Y$. Then the following are equivalent:

(1) Each S_α is left cancellative and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$;
(2) Each $\sum_\alpha$ is proper and $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$;
(3) Q is proper and $(\star)$ holds.

Proof:
(1) $\Longrightarrow$ (2): Since each S_α is left cancellative, it follows that $\sum_\alpha$ is proper for all $\alpha\in Y$, by Lemma 2.5. The implication

(2) $\Longrightarrow$ (3): follows from Theorem 3.1.

(3) $\Longrightarrow$ (1): Since Q is proper, it follows that each $\sum_\alpha$ is proper so that (1) follows from Lemma 1.5 and Theorem 3.1.

Remark 3.5: In the rest of this section let $S=\left[Y;S_\alpha;\varphi_{\alpha,\beta}\right]$ be a strong semilattice of right cancellative monoids $S_\alpha,\ \alpha\in Y$ with the (LC) condition, and assume that S has the (LC) condition, and, let $Q=[Y;\sum_\alpha;\phi_{\alpha,\beta}]$ be a semigroup of left I-quotients of S, where each $\sum_\alpha$ is the inverse hull of S_α for each $\alpha\in Y$. By Lemma 2.5 each S_α is a left I-order in $\sum_\alpha$ where S_α is the $\mathcal{R}$-class of the identity of $\sum_\alpha$.

Remark 3.6: From Lemma 2.5 and Lemma 2.3, we deduce that for any $a,b\in S_\alpha$ and for all $\alpha\in Y$ we have $a\ \mathcal{L}\ b$ in S_α implies that $a\ \mathcal{L}\ b$ in $\sum_\alpha$.

By the above Remark, $(\star)$ holds if and only if $(\star)'$ holds where $(\star)'$ is the following condition: for all $a,b\in S_\alpha$ and for all $\alpha\geq\beta$, $a\varphi_{\alpha,\beta}\ \mathcal{L}\ b\varphi_{\alpha,\beta}$ in S_β implies that $a\ \mathcal{L}\ b$ in S_α.

If we insisted on Q being proper, then by Theorem 3.1, the sufficient conditions are $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$ and $\sum_\alpha$ is proper for all $\alpha\in Y$. Such conditions are related to the structure of Q. We shall introduce equivalent conditions on the structure of S in order to do so. We begin with the following lemma.

Lemma 3.7: Let $\phi_{\alpha,\beta}$, $\varphi_{\alpha,\beta}$ and S be as in the Remark 3.5. If $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$, then

(*i*) $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$;
(*ii*) $(\star)'$ holds.

Proof:

(i) Since $\phi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$. Then as $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ we have that $\varphi_{\alpha,\beta}$ is one-to-one.
(ii) Suppose that $a\varphi_{\alpha,\beta}\ \mathcal{L}\ c\varphi_{\alpha,\beta}$ in S_β for all $\alpha\geq\beta$ in $Y$and $a,b\in S_\alpha$ so that $a\varphi_{\alpha,\beta}\ \mathcal{L}^{\Sigma_\beta}\ c\varphi_{\alpha,\beta}$, by Remark 3.6. Hence $a\phi_{\alpha,\beta}\ \mathcal{L}^{\Sigma_\beta}\ c\phi_{\alpha,\beta}$ so that $a\phi_{\alpha,\beta}{}^{-1}a\phi_{\alpha,\beta}=b\phi_{\alpha,\beta}{}^{-1}b\phi_{\alpha,\beta}$. We have that $(a^{-1}a)\phi_{\alpha,\beta}=(b^{-1}b)\phi_{\alpha,\beta}$. As $\phi_{\alpha,\beta}$ is one-to-one we have that $a^{-1}a=b^{-1}b$ so that $a\ \mathcal{L}^{\Sigma_\alpha}\ b$. By Remark 3.6, $a\ \mathcal{L}\ b$ in S_α as required.

Lemma 3.8: Let $\phi_{\alpha,\beta}$, $\varphi_{\alpha,\beta}$, S and Q be as in the Remark 3.5. Let Q be proper and $(\star)'$ holds. Then $\phi_{\alpha,\beta}$ is one-t-one if and only if $\varphi_{\alpha,\beta}$ is one-t-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$.

Proof: It is clear that if each $\phi_{\alpha,\beta}$ is one-to-one, then each $\varphi_{\alpha,\beta}$ is one-to-one. Conversely, suppose that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha,\beta\in Y$ with $\alpha\geq\beta$. Let

$$(a^{-1}b)\phi_{\alpha,\beta}=(c^{-1}d)\phi_{\alpha,\beta}$$

where $a^{-1}b,c^{-1}d\in\sum_\alpha$ for some $a,b,c,d\in S_\alpha$. Since $\phi_{\alpha,\beta}$ is the extension of $\varphi_{\alpha,\beta}$ for all $\alpha\geq\beta$ in Y, we have

$$\left(e_\beta a\right)^{-1}\left(e_\beta b\right)=\left(e_\beta c\right)^{-1}\left(e_\beta d\right).$$

Hence $a^{-1}e_\beta b=c^{-1}e_\beta d$ and so $a^{-1}be_\beta=c^{-1}de_\beta$ as the identities are central in Q. It follows that $a^{-1}b\ \sigma\ c^{-1}d$ in Q. By Lemma 2.4, $e_\beta a\ \mathcal{L}^{\Sigma_\beta}e_\beta b$ and so $e_\beta a\ \mathcal{L}\ e_\beta c$ in S_β, by Remark 3.6. Then $a\varphi_{\alpha,\beta}\ \mathcal{L}\ c\varphi_{\alpha,\beta}$ in S_β and so $a\ \mathcal{L}\ c$ in S_α, by $(\star)'$. Again by Remark 3.6, $a\ \mathcal{L}^{\Sigma_\alpha}\ c$. It follows that $a^{-1}\ \mathcal{R}^{\Sigma_\alpha}c^{-1}$ and so $a^{-1}\ \mathcal{R}^{Q}c^{-1}$, by Proposition 2.4.2 of [11]. Again by Lemma 2.4, $a^{-1}b\ \mathcal{R}^{Q}\ a^{-1}\mathcal{R}^{Q}c^{-1}\mathcal{R}^{Q}c^{-1}d$.

Since Q is proper we have that $a^{-1}b=c^{-1}d$, by Proposition 2.2. Thus $\phi_{\alpha,\beta}$ is one to one.

Before giving the conditions which make Q is proper, by using the structure of S we need the following lemma.

Lemma 3.9: Let $\varphi_{\alpha,\beta}$ and Q be as in the Remark 3.5. Then Q is proper if and only if $\sum_\alpha$ is proper for each $\alpha \in Y$ and $\varphi_{\alpha,\beta}$ is one-to-one, for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$.

Proof: Suppose that Q is proper. It is clear that $\sum_\alpha$ is proper for all $\alpha \in Y$. To show that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $a\varphi_{\alpha,\beta} = b\varphi_{\alpha,\beta}$ where $a, b \in S_\alpha$. Then $e_\beta a = e_\beta b$ so that $a\ \sigma\ b$ in. Since $a\ \mathcal{R}^{\Sigma_\alpha}\ b$, it follows that $a\ \mathcal{R}^Q\ b$, by Proposition 2.4.2 of [11]. As Q is proper we have that $a = b$, by Proposition 2.2. Thus $\varphi_{\alpha,\beta}$ is one to one.

On the other hand, assume that $\varphi_{\alpha,\beta}$ is one-to-one for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $\sum_\alpha$ is proper for all $\alpha \in Y$. Let

$$a^{-1}bc^{-1}c = c^{-1}c$$

where $a^{-1}b \in \sum_\alpha$ and $c^{-1}c \in \sum_\beta$ for some $a, b \in S_\alpha$ and $c \in S_\beta$. By definition of multiplication,

$$a^{-1}bc^{-1}c = (xa)^{-1}(yc) = c^{-1}c$$

where $xb = yc$ for some $x, y \in S_{\alpha\beta}$. It is clear that $\beta \leq \alpha$ so tha $S_{\alpha\beta} = S_\beta$. Hence $xa, yc, c \in S_\beta$ so that $xa = uc = yc$ for some unit u in S_β, by Lemma 2.3. Since $xb = yc$ we have that $xe_\beta b = yc = xe_\beta a$ and as $\sum_\beta$ is proper, we have that S_β is cancellative, by Lemma 2.5. Since $e_\beta a, e_\beta b$ and x are in S_β which is cancellative we obtain $e_\beta b = e_\beta a$. Hence $a\varphi_{\alpha,\beta} = b\varphi_{\alpha,\beta}$ gives $a = b$ as $\varphi_{\alpha,\beta}$ is one-to-one. Thus Q is proper.

From Lemma 2.5 and Lemma 3.9, we have

Corollary 3.10: Let S, $\varphi_{\alpha,\beta}$ and Q be as in the Remark 3.5. Then S_α is left cancellative for each $\alpha \in Y$ and $\varphi_{\alpha,\beta}$ is one-to-one, for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ if and only if Q is proper.

In the next lemma we shed light on the relationships between $\varphi_{\alpha,\beta}$, $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ for all $\alpha \geq \beta$ in Y which play a significant role in making Q proper.

Lemma 3.11: Let S, Q, $\varphi_{\alpha,\beta}$, $\phi_{\alpha,\beta}$ be as in the Remark 3.5 and $\psi_{\alpha,\beta}$ be as in Corollary 3.3. If S_α is left cancellative for all $\alpha \in Y$, then the following are equivalent:

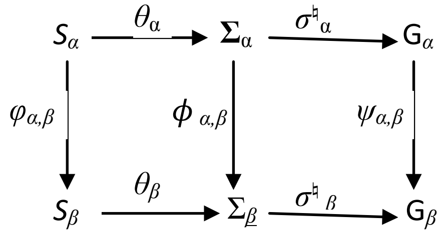
(i) Each $\varphi_{\alpha,\beta}$ is one-to-one and $(\star)'$ holds;
(ii) Each $\phi_{\alpha,\beta}$ is one-to-one and $(\star)$ holds;
(iii) Each $\psi_{\alpha,\beta}$ is one-to-one and $(\star)$ holds.

Proof: As S_α is left cancellative for each $\alpha \in Y$, we have that $\sum_\alpha$ is proper for each $\alpha \in Y$, by Lemma 2.5. Hence we can use Lemma 3.2.

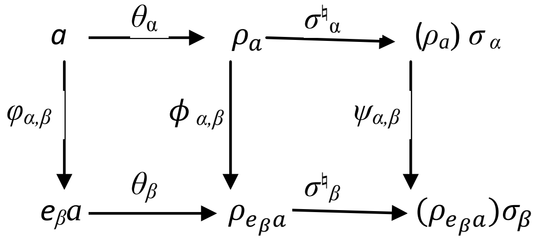
$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$ From Corollary 3.10, we have that Q is proper. Then (*ii*) follows from Theorem 3.1, and the equivalence between $(\star)$ and $(\star)'$.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{iii})$ By Lemma 3.4, we have that Q is proper and so (*iii*) follows from Lemma 3.2. The implication $(iii) \Longrightarrow (i)$ follows from Lemma 3.2, Theorem 3.1 and Remark 3.6.

The following diagrams may help the reader to visualise the relationship between the morphisms and semigroups which we have considered.



and



It is easy to see that the above diagrams are commutative.

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