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SEMIPRIME (-1, 1) RINGS

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ABSTRACT

In this paper, we show that in a (-1,1) ring R, every associator commutes with every element of R, that is ((R,R,R),R)=0 and (R,R (R,R,R))=0. Using these we prove that a 2- and 3- divisible semiprime (-1, 1) ring R is associative.

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1. INTRODUCTION

Thedy [1] studied nonassociative rings satisfying the identity ((a, b, c), d) = 0. He proved that a simple nonassociative ring with ((a, b, c,), d) = 0 is either associative or commutative. He pointed out that it cannot be extended to prime rings.

In this paper, we show that in a (-1,1) ring R, every associator commutes with every element of R, that is ((R, R, R), R) = 0 and (R, R (R, R, R)) = 0. Using these we prove that a 2- and 3- divisible semiprime (-1, 1) ring R is associative. At the end of this section we give an example of a (-1, 1) ring which is not associative.

2. PRELIMINARIES

A nonassociative ring is said to be a (-1, 1) ring if it satisfies the following identities:

$$A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{1}$$

and

$$B(x, y, z) = (x, y, z) + (x, z, y) = 0 \tag{2}$$

We know that a ring R is semi prime if for any ideal A of R, $A^2 = 0$ implies A = 0.

A ring R is said to be n – divisible if nx=0 implies x=0 for all x in R and n a natural number.

Throughout this section R denotes a 2- and 3- divisible (-1, 1) ring.

As a consequence of (2), we have the right alternative law (y, x, x) = 0. (3)

In any ring we have the following identities:

$$c(w, x, y, z) = (wx, y, z) – (w, xy, z) + (w, x, yz) - w(x, y, z) – (w, x, y)z = 0. \tag{4}$$

and

$$(xy, z) – x(y, z) – (x, z)y – (x, y, z) + (x, z, y) - (z, x, y) = 0. \tag{5}$$

By forming C(x, y, y, z) – C(x, z, y, y) + C(x, y, z, y) = 0,

we obtain 2(x, y, yz) = 2(x, y, z)y. This implies that

$$D(x, y, z) = (x, y, yz) – (x, y, z)y = 0. \tag{6}$$

In C(x, z, y, y) = 0 we make use of (6),

So that $E(x, y, z) = (x, y^2, z) – (x, y, yz + zy) = 0.$ (7)

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By linearizing (6) (replace y with w + y), we obtain the identity

$$F(x, w, y, z) = (x, w, yz) + (x, y, wz) – (x, w, z)y - (x, y, z)w = 0. \tag{8}$$

From C(w, x, y, z) – F (w, z, x, y) = 0, it follows that

$$G(w, x, y, z) = (wx, y, z) + (w, x, (y, z)) – w(x, y, z) – (w, y, z)x = 0.$$

In a (-1, 1) ring (5) becomes

$$H(x, y, z) = (xy, z) – x(y, z) – (x, z)y - 2(x, y, z) – (z, x, y) = 0,$$

Because of (2). The combination of (1) and (4) gives

$$J(w, x, y, z) = (w, (x, y, z)) – (x, (y, z, w)) + (y, (z, w, x)) – (z,(w, x, y)) = 0.$$

From J(x, x, x, y) + (x, B(x, y, x)) = 0, it follows that

$$2(x, (x, x, y)) = 0.$$

From this and the fact that (x, y, x) = - (x, x, y) we obtain

$$(x, (x, x, y)) = 0 \text{ and } (x, (x, y, x)) = 0. \tag{9}$$

Now J(y, x, y, x) = 0 gives 2(y, (x, y, x)) -2 (x, (y, x, y)) = 0.

Thus (y, (x, y, x)) – (x, (y, x, y)) = 0.

From B(x, x, y) = 0 and B(y, y, x) = 0, we have (y, (x, x, y)) - (x, (y, y, x)) = 0.

Combining this with J(y, x, x, y) = 0 gives 2(y, (x, x, y)) = 0 and therefore

$$(y, (x, x, y)) = 0. \tag{10}$$

Using the right alternative property of R, identity (10) can be written

$$(y, (x, y, x)) = 0. \tag{11}$$

Now we define U to be the set of all elements u of R which commute with all the elements of R.

That is, $U = \{u \in R/(u, R) = 0\}$.

Then C(x, x, u) = 0 gives – 2(x, x, u) = 0.

Hence (x, x, u) = 0 and (x, u, x) = 0 by (2).

Replacing x by x + y in these last two identities give

$$(x, y, u) = -(y, x, u) \tag{12}$$

and $$(x, u, y) = - (y, u, x), \text{ for } u \in U. \tag{13}$$

In addition to these identities, we present some more identities involving the element $u \in U$.

$$O = Q(u, x, y) = (u, x, y) – 2(y, x, u) \tag{14}$$

and $$O = R (x, y, u) = 3(x, y, u) – (x, y)u + (x, yu). \tag{15}$$

We know the identity (y, (x, y, x)) = 0, for every x, y, in R holds in R. Using this we prove the following lemma.

3. MAIN RESULTS

Lemma 1: If R is a 2- and 3- divisible (-1, 1) ring, then ((R, R, R), R) = 0.

Proof: Using the right alternative property (11) can be written as

$$(y, (x, x, y)) = 0. \tag{16}$$

By linearizing the identities (11) and (16), we have

$$(y, (x, y, z)) = - (y, (z, y, x)) \tag{17}$$

and $$(y, (x, z, y)) = - (y, (z, x, y)). \tag{18}$$

From equations (2), (17), (18) and again (2) we get

$$(y, (y, z, x)) = - (y, (y, x, z)) = (y, (z, x, y)) = -(y, (x, z, y)) = (y, (x, y, z)). \tag{19}$$

Community equation (1) with y, we have

$(y, (x, y, z) + (y, z, x) + (z, x, y)) = 0$. From (19)

This equation becomes $3(y,(x, y, z)) = 0$. Since R is 3- divisible,

$$(y, (x, y, z)) = 0. \tag{20}$$

From (20), the identity $L=(x, (y, y, z) – 3(y, (x, z, y)) = 0$ in [2] becomes $(x, (y, y, z)) = 0$.

Thus $$(R, (y, y, z)) = 0. \tag{21}$$

By linearizing equation (21), we obtain $(w, (x, y, z)) = – (w, (y, x, z))$. (22)

Applying equations (2) and (22) repeatedly, we get

$$(w, (x, y, z)) = - (w, (y, x, z)) = (w, (y, z, x)) = - (w, (z, y, x)) = (w, (z, x, y)).$$

Commuting equation (1) with w and applying the above equation, we obtain $3(w, (x, y, z)) = 0$.

Since R is 3- divisible, we have $(w, (x, y, z)) = 0$. (23)

This completes the proof of the lemma.

Lemma 2: If R is a 2 and 3 divisible (-1, 1) ring, then $(r, w(x, y, z)) = 0$.

Proof: Let r be an arbitrary element of R. By commuting equations (6), (8), (4) with r, and then applying (23) we get

$$(r, y(x, z, w) = -(r, w(x, z, y)), \tag{24}$$

$$(r, y(x, y, z)) = 0 \tag{25}$$

and $$(r, w(x, y, z))= –(r, z(w, x, y)). \tag{26}$$

Linearizing equation (25), we have

$$(r, w(x, y, z)) = -(r, y(x, w, z)). \tag{27}$$

Permutating cyclically (w z y x) in (26) and finally applying (24), we get

$$(r, w(x. y. z)) = - (r, z(w, x, y)) = (r, y(z, w, x)) = -(r, x(y, z, w)) = (r, w(y, z, x)). \tag{28}$$

But using (27) and $B(x, y, z) = 0$, (28) can be written as

$$(r, y(z, w, x)) = -(r, w(z, y, x)) = (r, w(z, x, y)). \tag{29}$$

Combining (28) and (29) we obtain

$$(r, w(x, y, z)) = (r, w(y, z, x)) = (r, w(z, x, y)). \tag{30}$$

Multiplying equation $A (x, y, z) = 0$ by w and commuting with r, and applying (30), then $3(r, w(x, y, z)) = 0$.

Since R is 3- divisible, we have $(r, w(x, y, z)) = 0$. (31)

Hence this completes the proof of the lemma.

Theorem 1: A 2- and 3-divisible semiprime (-1, 1) ring R is associative.

Proof: If u is an arbitrary associator, from (12) and (2) we have

$$(x, y, u) = -(y, x, u) = (y, u, x). \tag{32}$$

Using (3) and (32) we get

$$(u, x, y) = -(u, y, x) = -(y, x, u) = (y, u, x). \tag{33}$$

From (1) $(x, y, u) + (y, u, x) + (u, x, y) = 0$.

This implies $3(x, y, u) = 0$ using (32) and (33).

Therefore $(x, y, u) = 0$, since R is 3- divisible.

Associating equation (4) with r, s and using $(x, y, u) = 0$, then we obtain

$$\begin{aligned}(r, s, w(x, y, z)) &= - (r, s, (w, x, y,)z) \\ &= - (r, s,z(w, x, y)), \\ &= (r, s, (z, w, x)y), \text{ permutating z, w, x, y cyclically}\end{aligned}$$

= (r, s, y(z, w, x)),
= - (r, s, y(z, x, w)) using (2).
= (r, s, (y, z, x)w) again cyclically.
= (r, s, w(y, z, x).
= -(r, s, w(z, y, x)), using (21).
= (r, s, w(z, x, y)) using (2).

$\therefore$ (r, s, w(x, y, z)) = (r, s, w(y, z, x)) = (r, s, w(z, x, y)) (34)

Multiplying the equation (1) with w and associate with r, s then we obtain
(r, s, w(x, y, z)) + (r, s, w(y, z, x)) + (r, s, w(z, x, y)) = 0.

Using (34), the above equation becomes
3(r, s, w(x, y, z)) = 0, since R is 3- divisible then we have (r, s, w(x, y, z)) = 0.

We get (r, s, w) (x, y, z) = 0 by using (6).

Hence (R, R, R) (R, R, R) = 0.

We know that A is an associator ideal of R, so A.A=0, since R is semiprime then the ideal $A^2 = 0$ implies A=0.

That is (R, R, R) = 0. Hence R is associative.

Now we give an example of a(-1,1) ring, which is nonassociative.

Example: Consider the algebra having basis elements x, y and z over an arbitrary field. We define x^2=y, yx=z and all other products of basis elements equal to zero. It clearly satisfies (1) and (2) conditions. Hence it is a (-1, 1) ring, but not associative, since (x, x, x) = z.

4. REFERENCES

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