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INEQUALITIES CONCERNING THE INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS

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ABSTRACT

Let P (z) be a polynomial of degree n and P′(z) its derivative. In this paper we shall obtain an interesting generalization of De-Bruijn's Theorem and obtain as a special case the inequality due to Malik that if P(z) ≠ 0 for |z| < k, k ≥1, then $\underset{|z|=1}{Max}|P'(z)| \le \frac{n}{1+k}\underset{|z|=1}{Max}|P(z)|$ *and its generalization due to Govil.*

Keywords and Phrases: *Erdos-Lax Theorem, Maximum Modulus Principle, Zeros of a polynomial.*

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INTRODUCTION AND STATEMENT OF RESULTS:

Let P(z) be a polynomial of degree n and P′(z) its derivative, then

$$\underset{|z|=1}{Max}|P'(z)| \le n\,\underset{|z|=1}{Max}|P(z)| \qquad \text{and for } q \ge 1, \tag{1}$$

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le n\left\{\int_0^{2\pi}\left|P(e)^{i\theta}\right|^q d\theta\right\}^{1/q}. \tag{2}$$

Inequality (1) is an immediate consequence of S. Bernstein's Theorem on the derivative of a trigonometric polynomial (for reference see [9], [10] and [11]). Inequality (2) is due to Zygmund [12] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form P(e$^{i\square}$).Inequality (1) can be obtained by letting $q \to \infty$ in the inequality (2). Both the inequalities (1) and (2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in |z| < 1. In this connection it was conjectured by P. Erdos and later verified by Lax [7] (for other proof see [2]) that if P(z) does not vanish in |z| < 1, then inequality (1) can be replaced by[1]

Theorem 1.1: If P (z) = $\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in |z| < 1, then

$$\underset{|z|=1}{\mathrm{Max}}|P'(z)| \le \frac{n}{2}\underset{|z|=1}{\mathrm{Max}}|P(z)| \tag{3}$$

Equality in (3) holds if all zeros of P (z) lie in |z| = 1. This result was extended by Malik [8] who proved.

Theorem 1.2: If P (z) = $\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which has no zero in the disk |z| < k, k ≥ 1,

$$\underset{|z|=1}{Max}|P'(z)| \le \frac{n}{1+k}\underset{|z|=1}{Max}|P(z)|\ . \tag{4}$$

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The result is best possible and equality holds for P (z) = $(z + k)^n$.

As a refinement of Theorem 1.1 Aziz and Dawood [1] have shown that

Theorem 1.3: If P (z) = $\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in the disk |z| < 1, then

$$\underset{|z|=1}{\mathrm{Max}}|P'(z)| \le \frac{n}{2}\left\{\underset{|z|=1}{\mathrm{Max}}|P(z)| - \underset{|z|=1}{\mathrm{Min}}|P(z)|\right\} \tag{5}$$

The result is best possible and equality in (5) holds for the polynomial P (z) = $\alpha z^n + \beta$, where $|\beta| \ge |\alpha|$.

Theorem 1.3 was generalized by Govil [6] who proved the following result:

Theorem 1.4: If P (z) = $\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in |z| = 1, k ≥ 1, then

$$\underset{|z|=k}{\mathrm{Max}}|P'(z)| \le \frac{n}{1+k}\underset{|z|=1}{\mathrm{Max}}|P(z)| - \frac{n}{1+k}\underset{|z|=k}{\mathrm{Min}}|P(z)| \tag{6}$$

De-Bruijn [5] found out the following refinement of inequality (2).

Theorem 1.5: If P (z) is a polynomial of degree n which has no zeros in the disk |z| < 1, then for p ≥ 1,

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^p d\theta\right\}^{1/p} \le nC_p\left\{\int_0^{2\pi}\left|P(e)^{i\theta}\right|^p d\theta\right\}^{1/p}, \tag{7}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha\right\}^{-1/p}$$

The result is best possible and equality in (7) holds for P (z) = $\alpha z^n + \beta$, where $|\beta| \ge |\alpha|$.

The case p = 2 was obtained by Lax [7], where as, if we let $p \to \infty$ in (7) we get Erdos – Lax Theorem (Theorem 1.1).

In this paper we shall present the following result which is an interesting generalization of Theorem 1.5 and includes as a special case Theorem 1.2 due to Malik [8] and its generalization due to Govil [6].

Theorem 1.6: If P (z) = $\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in |z| < k, where k ≥ 1 and m = $\underset{|z|=k}{\mathrm{Min}}|p(z)|$, then for every real or complex number β with |β| ≤ 1, and for every p > 0, we have

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)+\frac{mn\beta}{1+k}\right|^p d\theta\right\}^{\frac{1}{p}} \le nC_p\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \tag{8}$$

where

$$\mathrm{Cp} = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|k+e^{i\alpha}\right|^p d\alpha\right\}^{\frac{-1}{p}}$$

Remark: Letting $p \to \infty$ in (8) and choosing argument of β with $|\beta| = 1$ suitably, it follows that

$$\underset{|z|=1}{\mathrm{Max}}|p'(z)| + \frac{mn}{1+k} \le \frac{n}{1+k}\underset{|z|=1}{\mathrm{Max}}|p(z)|.$$

If we take k = 1 in Theorem 1.6, we get the following interesting refinement of De Bruijn’s Theorem (Theorem 1.5) for p>0

Corollary: If P (z) is a polynomial of degree n which does not vanish in $|z|<1$ and m = $\underset{|Z|=1}{Min}|P(z)|$, then for every real or complex β with |β| ≤ 1 and for every p>0, we have

$$\left\{\int_0^{2\pi}\left|p'(e^{i\theta})+\frac{mn\beta}{2}\right|^p d\theta\right\}^{\frac{1}{p}} \le nC_p\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}, \tag{9}$$

where.

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha\right\}^{\frac{-1}{p}}$$

LEMMAS:

For the proof of Theorem 1.6, we need the following Lemmas.

Lemma: 2.1. If P (z) = $a_0 + \sum_{j=m}^{n} a_j z^j$ has no zeros in $|z|\le k$, k ≥ 1, then

$$k^m\left|P'(z)\right| \le \left|Q'(z)\right| \qquad \text{for} \qquad |z| = 1$$

and

$$\underset{|z|=1}{Max}\left|P'(z)\right| \le \frac{n}{1+k^m}\underset{|z|=1}{Max}\left|P(z)\right| \tag{10}$$

where

$$Q(z) = z^n\overline{P\left(\frac{1}{\overline{z}}\right)} \tag{11}$$

which is due to Chan and Malik [4].

Lemma: 2.2. If P (z) is a polynomial of degree n, then for every real α and for every p > 0,

$$\int_0^{2\pi}\left|nP\left(e^{i\theta}\right)-\left(1-e^{i\theta}\right)e^{i\alpha}P'\left(e^{i\theta}\right)\right|^p d\theta \le n^p\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta \tag{12}$$

Lemma 2.2 is due to Melas ([9] Inequality 5).

The following Lemma which is of independent interest is also needed for the proof of Theorem 1.6.

Lemma: 2.3. If A, B and C are non-negative real numbers such that B + C ≤ A then for every real α, 0 ≤ α < 2π, we have

$$\left|(A-C)+(B+C)e^{i\alpha}\right| \le \left|A+Be^{i\alpha}\right| \tag{13}$$

Proof of Lemma 2.3: If C = 0, then Lemma 2.3 is obvious. So we suppose C > 0. Since cos α ≤ 1 for all real α and by hypothesis A – B – C ≥ 0, it follows that

$$(A - B - C)\cos\alpha \le (A - B - C).$$

Multiplying both sides of this inequality by 2C and noting that C > 0, we get

$$\{2C(A - B) - 2C^2\}\cos\alpha \le 2C(A - B - C).$$

or equivalently,

$$2\{C(A - B) - C^2\}\cos\alpha + 2C^2 - 2C(A - B) \le 0.$$

Adding $A^2 + B^2 + 2AB\cos\alpha$ both sides and rearranging the terms, we get

$$(A^2 - 2AC + C^2)+(B^2 - 2BC + C^2) + 2(A-C)(B + C)\cos\alpha \le A^2 + B^2 + 2AB\cos\alpha.$$

which implies,

$$\left|(A-C)+e^{i\alpha}(B+C)\right|^2 \le \left|A+e^{i\alpha}B\right|^2$$

and hence

$\left|(A-C)+e^{i\alpha}(B+C)\right| \le \left|A+e^{i\alpha}B\right|,$ for every α, which is (13). This completes the proof of the Lemma 2.3.

Proof of Theorem: 1.6. By hypothesis, the polynomial p(z) has all its zeros in $|z| \ge k$, k ≥ 1, and m = $\underset{|z|=k}{\mathrm{Min}}|p(z)|$, therefore m ≤ $|p(z)|$ for $|z| \le k$. We show for any given complex number α with $|z| \le 1$, the polynomial F(z) = P(z) + α m has all its zeros in $|z| \ge k$. This is obvious if m = 0 that is if p(z) has a zero on $|z| = k$. We now suppose all the zeros of p(z) lie in $|z| > k$ so that m = $\underset{|z|=k}{\mathrm{Min}}|p(z)| > 0$.

Hence P(z)is analytic for $|z| = k$ and $\left|\frac{m}{P(z)}\right| \le 1$ for $|z| \le k$. Moreover $\frac{m}{p(z)}$ is not a constant therefore, it follows by Minimum Modulus Principle that

$$m < |P(z)| \qquad \text{for} \qquad |z| = k. \tag{14}$$

We assume that F (z) = P(z) + α m has a zero in $|z| < k$, say z = z_0 with $|z_0| < k$, then

$$P(z_0) + \alpha\, m = F(z_0) = 0.$$

This implies,

$$|P(z_0)| = |\alpha m| \le m,$$

which is a contradiction to (14). Hence we conclude that in any case F (z) = P(z) + αm has all its zeros in $|z| \ge k$. Applying Lemma 2.1 with m = 1 to the polynomial F(z), we get

$$K|F'(z)| \le |G'(z)| \tag{15}$$

where,

$$G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)} = Z^n \overline{P\left(\frac{1}{\bar{z}}\right)} - \bar{\alpha} z^n m$$

$$= Q(z) - \bar{\alpha} z^n m$$

Using F′(z) = P′(z) and G′(z) = Q′(z) - $n\bar{\alpha}z^{n-1}m$ in (15), we have

$$K|P'(z)| \le \left|Q'(z) - n\bar{\alpha} m Z^{n-1}\right| \quad \text{for} \quad |z| = 1. \tag{16}$$

Since all the zeros of G(z) = Q(z) - $\bar{\alpha}$ m Z^n lie in $|z| \le \frac{1}{k} \le 1$, by Gauss – Lucas Theorem, it follows that all the zeros of G′(z) = Q′(z)- $\bar{\alpha}$ nZ^{n-1}m also lie in $|z| \le \frac{1}{k} \le 1$ for every α with $|\alpha| \le 1$. This implies

$$Q'(z) \ge mn|Z|^{n-1} \quad \text{for} \quad |z| \ge \frac{1}{k}.$$

In particular,

$$|Q'(z)| \ge mn \qquad \text{for} \qquad |z| = 1 \tag{17}$$

Choosing argument of α with $|\alpha| = 1$ in the R.H.S of (16) such that

$$\left|Q'(z) - n\bar{\alpha} m z^{n-1}\right| = |Q'(z)| - nm \qquad \text{for} \qquad |z| = 1$$

which is possible by (17), therefore

$$k|p'(z)| \le |Q'(z)| + mn \qquad \text{for} \quad |z| = 1.$$

$$k\left\{|p'(z)| + \frac{mn}{1+k}\right\} \le |Q'(z)| - \frac{mn}{1+k} \qquad \text{for } |z| = 1. \tag{18}$$

Since it can be easily verified that

$$|np(z) - zp'(z)| = |Q'(z)| \qquad \text{for} \quad |z| = 1,$$

it follows from (18) that for each θ, 0≤ θ ≤ 2π, we have

$$k\left\{|p'(e^{i\theta})| + \frac{mn}{1+k}\right\} \le |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| - \frac{mn}{1+k}. \tag{19}$$

Applying Lemma 2.3, with

$A = |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})|$; $B = |p'(e^{i\theta})|$ and $C = \dfrac{mn}{1+k}$ and noting that by (19),

B + C ≤ A, we get

$$\left\{\left|\left(|np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| - \frac{mn}{1+k}\right) + \left(\left|p'(e^{i\theta}) + \frac{mn}{1+k}\right|e^{i\alpha}\right)\right|\right\} \le \left||np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| + e^{i\alpha}|p'(e^{i\theta})|\right|.$$

Hence for every p > 0, we have

$$\int_0^{2\pi}\left|\left\{|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| - \frac{mn}{1+k}\right\} + \left\{|P'(e^{i\theta})| + \frac{mn}{1+k}\right\}(e^{i\alpha})\right|^p d\alpha \le \int_0^{2\pi}\left||nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + e^{i\alpha}|P'(e^{i\theta})|\right|^p d\alpha$$

$$= \int_0^{2\pi}\left||nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + e^{i\alpha}|e^{i\theta}P'(e^{i\theta})|\right|^p d\alpha$$

$$\int_0^{2\pi}\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + e^{i\alpha}e^{i\theta}P'(e^{i\theta})\right|^p d\alpha. \tag{20}$$

Integrating both sides of (20) w. r. t θ from 0 to2π and using Lemma 2.2, we get

$$\int_0^{2\pi}\int_0^{2\pi}\left|\left\{|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| - \frac{mn}{1+k}\right\} + \left\{|P'(e^{i\theta})| + \frac{mn}{1+k}\right\}e^{i\alpha}\right|^p d\alpha d\theta \tag{21}$$

$$\le \int_0^{2\pi}\left\{\int_0^{2\pi}\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + e^{i\alpha}e^{i\theta}P'(e^{i\theta})\right|^p d\theta\right\}d\alpha$$

$$\le \int_0^{2\pi} n^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta d\alpha = 2\pi n^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta..$$

But

$$\int_0^{2\pi}\left|\left\{\left|nP\left(e^{i\theta}\right)-e^{i\theta}P'\left(e^{i\theta}\right)\right|-\frac{mn}{1+k}\right\}+\left\{\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k}\right\}e^{i\alpha}\right|^p d\alpha$$

$$=\left|\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k}\right|^p\int_0^{2\pi}\left|e^{i\alpha}+\frac{\left|nP\left(e^{i\theta}\right)-e^{i\theta}P'\left(e^{i\theta}\right)\right|-\frac{mn}{1+k}}{\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k}}\right|^p d\alpha$$

$$\geq\left\{\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k}\right\}^p\int_0^{2\pi}\left|e^{i\alpha}+k\right|^p d\alpha. \qquad \text{(Using (19))}$$

Using this in (21), we get for each p > o,

$$\int_0^{2\pi}\left|\left|P'\left(e^{i\theta}\right)\right|+\frac{mn}{1+k}\right|^p d\theta\int_0^{2\pi}\left|e^{i\alpha}+k\right|^p d\alpha\leq 2\pi n^p\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta,.$$

From which the desired result follows immediately and this completes the proof of the Theorem 1.6.

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