



Fuzzy pgprw-open maps and fuzzy pgprw-closed maps in fuzzy topological spaces

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ABTRACT

In this paper, we introduce fuzzy pgprw-open and fuzzy pgprw-closed maps in fts and obtain certain characterization of the pgprw closed and pgprw open maps.

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INTRODUCTION

The concept of a fuzzy subset was introduced and studied by L.A.Zadeh in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of science and engineering. In the year 1965, L.A.Zadeh [1] introduced the concept of fuzzy subset as a generalization of that of an ordinary subset. The introduction of fuzzy subsets paved the way for rapid research work in many areas of mathematical science. In the year 1968, C.L.Chang [2] introduced the concept of fuzzy topological spaces as an application of fuzzy sets to topological spaces. Subsequently several researchers contributed to the development of the theory and applications of fuzzy topology. The theory of fuzzy topological spaces can be regarded as a generalization theory of topological spaces. An ordinary subset A or a set X can be characterized by a function called characteristic function

$$\mu_A : X \rightarrow [0,1] \text{ of } A, \text{ defined by}$$
$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A. \\ 0, & \text{if } x \notin A. \end{cases}$$

Thus an element $x \in X$ is in A if $\mu_A(x) = 1$ and is not in A if $\mu_A(x) = 0$. In general if X is a set and A is a subset of X then A has the following representation. $A = \{(x, \mu_A(x)) : x \in X\}$, here $\mu_A(x)$ may be regarded as the degree of belongingness of x to A, which is either 0 or 1. Hence A is the class of objects with degree of belongingness either 0 or 1 only. Prof. L.A.Zadeh [1] introduced a class of objects with continuous grades of belongingness ranging between 0 and 1 ; he called such a class as fuzzy subset.

A fuzzy subset A in X is characterized as a membership function $\mu_A : X \rightarrow [0,1]$, which associates with each point in x a real number $\mu_A(x)$ between 0 and 1 which represents the degree or grade membership of belongingness of x to A.

The purpose of this paper is to introduce a new class of fuzzy sets called fuzzy pgprw-closed sets in fuzzy topological spaces and investigate certain basic properties of these fuzzy sets. Among many other results it is observed that every fuzzy closed set is fuzzy pgprw-closed but not conversely. Also we introduce fuzzy pgprw-open sets in fuzzy topological spaces and study some of their properties.

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1. PRELIMINARIES

1.1 Definition:[1] A fuzzy subset A in a set X is a function $A: X \rightarrow [0, 1]$. A fuzzy subset in X is empty iff its membership function is identically 0 on X and is denoted by 0 or μ_ϕ . The set X can be considered as a fuzzy subset of X whose membership function is identically 1 on X and is denoted by μ_x or I_x . In fact every subset of X is a fuzzy subset of X but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

1.2 Definition: [1] If A and B are any two fuzzy subsets of a set X , then A is said to be included in B or A is contained in B iff $A(x) \leq B(x)$ for all x in X . Equivalently, $A \leq B$ iff $A(x) \leq B(x)$ for all x in X .

1.3 Definition: [1] Two fuzzy subsets A and B are said to be equal if $A(x) = B(x)$ for every x in X . Equivalently $A = B$ if $A(x) = B(x)$ for every x in X .

1.4 Definition: [1] The complement of a fuzzy subset A in a set X , denoted by A' or $1 - A$, is the fuzzy subset of X defined by $A'(x) = 1 - A(x)$ for all x in X . Note that $(A')' = A$.

1.5 Definition: [1] The union of two fuzzy subsets A and B in X , denoted by $A \vee B$, is a fuzzy subset in X defined by $(A \vee B)(x) = \text{Max}\{\mu_A(x), \mu_B(x)\}$ for all x in X .

1.6 Definition: [1] The intersection of two fuzzy subsets A and B in X , denoted by $A \wedge B$, is a fuzzy subset in X defined by $(A \wedge B)(x) = \text{Min}\{A(x), B(x)\}$ for all x in X .

1.7 Definition: [1] A fuzzy set on X is 'Crisp' if it takes only the values 0 and 1 on X .

1.8 Definition: [2] Let X be a set and τ be a family of fuzzy subsets of X . τ is called a fuzzy topology on X iff τ satisfies the following conditions.

(i) $\mu_\phi; \mu_x \in \tau$: That is 0 and $1 \in \tau$

(ii) If $G_i \in \tau$ for $i \in I$ then $\bigvee_{i \in I} G_i \in \tau$

(iii) If $G, H \in \tau$ then $G \wedge H \in \tau$

The pair (X, τ) is called a fuzzy topological space (abbreviated as fts). The members of τ are called fuzzy open sets and a fuzzy set A in X is said to be closed iff $1 - A$ is a fuzzy open set in X .

1.9 Remark: [2] Every topological space is a fuzzy topological space but not conversely.

1.10 Definition: [2] Let X be a fts and A be a fuzzy subset in X . Then $\bigwedge \{B: B \text{ is a closed fuzzy set in } X \text{ and } B \geq A\}$ is called the closure of A and is denoted by A or $\text{cl}(A)$.

1.11 Definition: [2] Let A and B be two fuzzy sets in a fuzzy topological space (X, τ) and let $A \geq B$. Then B is called an interior fuzzy set of A if there exists $G \in \tau$ such that $A \geq G \geq B$, the least upper bound of all interior fuzzy sets of A is called the interior of A and is denoted by A^0 .

1.12 Definition: [3] A fuzzy set A in a fts X is said to be fuzzy semiopen if and only if there exists a fuzzy open set V in X such that $V \leq A \leq \text{cl}(V)$.

1.13 Definition: [3] A fuzzy set A in a fts X is said to be fuzzy semi-closed if and only if there exists a fuzzy closed set V in X such that $\text{int}(V) \leq A \leq V$. It is seen that a fuzzy set A is fuzzy semiopen if and only if $1 - A$ is a fuzzy semi-closed.

1.14 Theorem: [3] The following are equivalent:

- (a) μ is a fuzzy semiclosed set,
- (b) μ^c is a fuzzy semiopen set,
- (c) $\text{int}(\text{cl}(\mu)) \leq \mu$.
- (b) $\text{int}(\text{cl}(\mu)) \geq \mu^c$

1.15 Theorem [3] Any union of fuzzy semiopen sets is a fuzzy semiopen set and (b) any intersection of fuzzy semi closed sets is a fuzzy semi closed.

1.16 Remark: [3]

- (i) Every fuzzy open set is a fuzzy semiopen but not conversely.
- (ii) Every fuzzy closed set is a fuzzy semi-closed set but not conversely.
- (iii) The closure of a fuzzy open set is fuzzy semiopen set
- (iv) The interior of a fuzzy closed set is fuzzy semi-closed set

1.17 Definition: [3] A fuzzy set μ of a fts X is called a fuzzy regular open set of X if $\text{int}(\text{cl}(\mu)) = \mu$.

1.18 Definition: [3] A fuzzy set μ of fts X is called a fuzzy regular closed set of X if $\text{cl}(\text{int}(\mu)) = \mu$.

1.19 Theorem: [3] A fuzzy set μ of a fts X is a fuzzy regular open if and only if μ^c fuzzy regular closed set.

1.20 Remark: [3]

- (i) Every fuzzy regular open set is a fuzzy open set but not conversely.
- (ii) Every fuzzy regular closed set is a fuzzy closed set but not conversely.

1.21 Theorem: [3]

- (i) The closure of a fuzzy open set is a fuzzy regularclosed.
- (ii) The interior of a fuzzy closed set is a fuzzy regular open set.

1.22 Definition: [4] A fuzzy set α in fts X is called fuzzy rwclosed if $\text{cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is regular semi-open in X.

1.23 Definition: [5] A fuzzy set α in fts X is called fuzzy pgprw closed if $\text{p-cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is $\text{rg}\alpha$ -open set in X.

1.24 Defintion: [5] A fuzzy set α of a fts X is fuzzy pgprw-open set, if it's complement α^c is a fuzzy pgprw-closed in fts X.

1.25 definition: [2] Let X and Y be fts. A map $f: X \rightarrow Y$ is said to be a fuzzy continuous mapping if $f^{-1}(\mu)$ is fuzzy open in X for each fuzzy open set μ in Y.

1.26 definition: [6] A mapping $f: X \rightarrow Y$ is said to be fuzzy irresolute iff $f^{-1}(B)$ is fuzzy semi-open in X is fuzzy open in Y.

1.27 definition: [7] A mapping $f: X \rightarrow Y$ is said to be fuzzy pgprw irresolute if the inverse image of every fuzzy pgprw-open in Y is a fuzzy pgprw-open set in X.

2. FUZZY PGPRW-OPEN MAPS AND FUZZY PGPRW-CLOSED MAPS IN FUZZY TOPOLOGICAL SPACES

Definition 2.1: Let X and Y be two fts. A map $f: (X, T_1) \rightarrow (Y, T_2)$ is called fuzzy pgprw-open map if the image of every fuzzy open set in X is fuzzy pgprw-open in Y.

Theorem 2.2: Every fuzzy open map is a fuzzy pgprw-open map.

Proof: Let $f: (X, T_1) \rightarrow (Y, T_2)$ be a fuzzy open map and μ be a fuzzy open set in fts X. Then $f(\mu)$ is a fuzzy open set in fts Y. Since every fuzzy open set is fuzzy pgprw-open, $f(\mu)$ is a fuzzy pgprw-open set in fts Y. Hence f is a fuzzy pgprw-open map.

The converse of the above theorem need not be true in general as seen from the following example.

Example 2.3: Let $X = Y = \{a, b, c\}$ and the functions $\alpha, \beta: X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = a. \\ 0, & \text{otherwise} \end{cases}$$

Consider $T_1 = \{1, 0, \alpha\}$, $T_2 = \{1, 0, \beta\}$ then $(X, T_1) \rightarrow (Y, T_2)$ are fts. Let map $f: X \rightarrow Y$ be the identity map. Then this function is pgprw-open map but it is not fuzzy open .Since the image of the fuzzy open set $\alpha(x)$ in X is fuzzy set β in Y which is not fuzzy open.

Remark 2.4: If $f: (X, T_1) \rightarrow (Y, T_2)$ and $g: (Y, T_2) \rightarrow (Z, T_3)$ be two fuzzy pgprw—open map then composition $g \circ f: (X, T_1) \rightarrow (Z, T_3)$ need not be fuzzy pgprw-open as seen from the following example

Example 2.5: Let $X=Y=Z=\{a,b,c\}$ and the function $\alpha, \beta, \gamma, \delta: X \rightarrow [0,1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}$$

Consider $T_1 = \{1,0,\alpha, \delta\}$, $T_2 = \{1,0,\alpha\}$ and $T_3 = \{0,1, \alpha, \beta, \gamma\}$. Let map $f: (X, T_1) \rightarrow (Y, T_2)$ and $g: (Y, T_2) \rightarrow (Z, T_3)$ be the identity maps then f and g are fuzzy pgprw-open map but their composition $g \circ f: (X, T_1) \rightarrow (Z, T_3)$ is not fuzzy pgprw-open as $\beta: X \rightarrow [0,1]$ is fuzzy open in X But $(g \circ f)(\beta) = \beta$ is not fuzzy pgprw-open in Z .

Theorem 2.6: If $f: (X, T_1) \rightarrow (Y, T_2)$ is fuzzy open map and $g: (Y, T_2) \rightarrow (Z, T_3)$ is fuzzy pgprw—open map then their composition $g \circ f: (X, T_1) \rightarrow (Z, T_3)$ is fuzzy pgprw-open map.

Proof: Let α be fuzzy open set in (X, T_1) . since f is fuzzy open map, $f(\alpha)$ is a fuzzy open set in (Y, T_2) . Since g is a fuzzy pgprw-open map $g(f(\alpha)) = (g \circ f)(\alpha)$. Thus $g \circ f$ is a fuzzy pgprw-open map.

Definition 2.7: Let X and Y be two fuzzy topological spaces. A map $f: (X, T_1) \rightarrow (Y, T_2)$ is called fuzzy pgprw-closed map if the image of every fuzzy closed set in X is a fuzzy closed set in Y .

Theorem 2.8: Let $f: (X, T_1) \rightarrow (Y, T_2)$ be fuzzy closed map then f is a fuzzy pgprw closed map.

Proof: Let α be a fuzzy closed set in (X, T_1) . since f is fuzzy closed map, $f(\alpha)$ is a fuzzy closed set in (Y, T_2) . since every fuzzy closed set is fuzzy pgprw-closed, $f(\alpha)$ is a fuzzy pgprw-closed set in (Y, T_2) . Hence f is fuzzy pgprw-closed map.

The converse of the theorem need not be true in general as seen from the following example

Example 2.9: Let $X= Y=\{a, b, c\}$ and the functions $\alpha, \beta: X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

Consider $T_1 = \{1,0,\alpha\}$ and $T_2 = \{1,0,\beta\}$; then (X, T_1) and (Y, T_2) are fits. Let $f: (X, T_1) \rightarrow (Y, T_2)$ be the identity map. Then f is a fuzzy pgprw-closed map but it is not a fuzzy closed map, since the image of the closed set α in X is not a fuzzy closed set in Y .

Remark 2.10: The composition of two fuzzy pgprw-closed maps need not be a fuzzy pgprw-closed map

Example 2.11: Consider the fits $(X, T_1), (Y, T_2), (Z, T_3)$ and mappings defined in 2.5. the maps f and g are fuzzy pgprw-closed but their composition is not fuzzy pgprw-closed as $\beta: X \rightarrow [0,1]$ is a fuzzy closed set in X but $(g \circ f)(\beta) = \beta$ is not fuzzy pgprw-closed in Z .

Theorem 2.12: If a map $f: (X, T_1) \rightarrow (Y, T_2)$ and $g: (Y, T_2) \rightarrow (Z, T_3)$ be two maps then composition of two pgprw-closed maps is fuzzy pgprw-closed map.

Proof: Let α be a fuzzy closed set in (X, T_1) . Since f is a fuzzy closed map, $f(\alpha)$ is a fuzzy closed set in (Y, T_2) . Since g is a fuzzy pgprw-closed map but $g(f(\alpha))$ is a fuzzy pgprw-closed set in (Z, T_3) but $g(f(\alpha)) = (g \circ f)(\alpha)$. Thus $g \circ f$ is fuzzy pgprw-closed map.

Theorem 2.13: A map $f: X \rightarrow Y$ is fuzzy pgprw-closed map if for each fuzzy set δ of Y and for each fuzzy open set μ of X s.t $\mu \geq f^{-1}(\delta)$, there is a fuzzy pgprw-open map set α of Y s.t $\delta \leq \alpha$ and $f^{-1}(\alpha) \leq \mu$.

Proof: Suppose that f is fuzzy pgprw-closed map. Let δ be a fuzzy subset of Y and μ be a fuzzy open set of X s.t $f^{-1}(\delta) \leq \mu$. Let $\alpha = 1 - f(1 - \mu)$ is fuzzy pgprw-open set in Y . note that $f^{-1}(\delta) \leq \mu$. Which implies $\delta \leq \alpha$ and $f^{-1}(\alpha) \leq \mu$.

For the converse, Suppose that μ is a fuzzy closed set in X , Then $f^{-1}(1 - f(\mu)) \leq 1 - \mu$

And $1 - \mu$ is fuzzy open by hypothesis, there is a fuzzy pgprw-open set α of Y s.t $(1 - f(\mu)) \leq \alpha$.

And $f^{-1}(\alpha) \leq 1 - \mu$. Therefore $\mu \leq 1 - f^{-1}(\alpha)$. Hence $1 - \alpha \leq \mu$, $f(1 - f^{-1}(\alpha)) \leq 1 - \alpha$.

Which implies $f(\mu) = 1 - \alpha$. since $1 - \alpha$ is fuzzy pgprw-closed $f(\mu)$ is fuzzy pgprw-closed and thus f is fuzzy pgprw-closed.

Lemma 2.14: Let $f: (X, T_1) \rightarrow (Y, T_2)$ be fuzzy irresolute map and α be fuzzy $rg\alpha$ open in Y then $f^{-1}(\alpha)$ is fuzzy $rg\alpha$ open in X .

Proof: Let α be fuzzy $rg\alpha$ -open in Y . To prove $f^{-1}(\alpha)$ is fuzzy $rg\alpha$ open in X that is to prove $f^{-1}(\alpha)$ is both fuzzy semi-open and fuzzy semi-closed in X . Now α is fuzzy semi-open in Y . Since f is fuzzy irresolute map, $f^{-1}(\alpha)$ is fuzzy semi-open in X .

Now α is fuzzy semi-closed in Y as fuzzy $rg\alpha$ open set is fuzzy semi-closed then $1 - \alpha$ is Fuzzy semi-open in Y . Since f is fuzzy irresolute map, $f^{-1}(1 - \alpha)$ is a fuzzy semi open in X . But $f^{-1}(1 - \alpha) = 1 - f^{-1}(\alpha)$ is fuzzy semi-open in X and so $f^{-1}(\alpha)$ is semi closed in X . Thus $f^{-1}(\alpha)$ is both fuzzy semi-open and fuzzy semi-closed in X and hence $f^{-1}(\alpha)$ is fuzzy $rg\alpha$ open in X .

Theorem 2.15: If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is fuzzy irresolute map and fuzzy pgprw-closed and α is fuzzy pgprw-closed set of X , then $f(\alpha)$ is a fuzzy pgprw-closed set in Y .

Proof: Let α be a fuzzy closed set of X . Let $f(\alpha) \leq \mu$. Where μ is fuzzy $rg\alpha$ open in Y . Since f is fuzzy irresolute map, $f^{-1}(\mu)$ is a fuzzy $rg\alpha$ open in X , by lemma 2.14 and $\alpha \leq f^{-1}(\mu)$. Since α is a fuzzy pgprw-closed set in X , $P-cl(\alpha) \leq f^{-1}(\mu)$. Since f is a fuzzy pgprw-closed set contained in the fuzzy $rg\alpha$ open set μ , which implies $cl(f(P-cl(\alpha))) \leq \mu$ and hence $cl(f(\alpha)) \leq \mu$. Therefore $f(\alpha)$ is a fuzzy pgprw-closed set in Y .

Corollary 2.16: If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is fuzzy irresolute map and fuzzy closed and α is a fuzzy pgprw-closed set in X .

Proof : The proof follows from the theorem 2.15 and the fact that every fuzzy closed map is a fuzzy pgprw-closed map.

Theorem 2.17: Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two mappings s.t

- (i) if f is fuzzy continuous map and surjective, then g is fuzzy pgprw-closed map.
- (ii) if g is fuzzy pgprw-irresolute map and injective then f is fuzzy pgprw-closed map.

Proof:

- (i) Let μ be a fuzzy closed set in Y , since f is fuzzy continuous map, $f^{-1}(\mu)$ is a fuzzy closed set in X . Since $g \circ f$ is a fuzzy pgprw closed map, $(g \circ f)(f^{-1}(\mu))$ is a fuzzy pgprw-closed set in Z . but $(g \circ f)(f^{-1}(\mu)) = g(\mu)$ as f is surjective thus g is fuzzy pgprw closed map.
- (ii) Let β be a fuzzy closed set of X then $(g \circ f)(\beta)$ is a fuzzy pgprw closed set in Z , Since $(g \circ f)$ is a fuzzy pgprw-closed map. Since g is fuzzy pgprw-irresolute map $g^{-1}(g \circ f)(\beta)$ is fuzzy pgprw-closed in Y . But $g^{-1}(g \circ f)(\beta) = f(\beta)$ as g is injective. Thus f is fuzzy pgprw-closed map.

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