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A SHORT STUDY ON DIFFERENTIAL SUPERORDINATION

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ABSTRACT

In this paper, the differential superordination for certain classes of univalent and analytic functions has been discussed. Some various properties and results on the differential superordination have been summarized.

Keywords: *Analytic function, Univalent function, Superordination, Differential Superordination, Admissible Functions.*

1. INTRODUCTION

Recently, much attention has been paid to study the concept of the subordination and superordination in the theory of analytic function. Significant and interesting problems in that theory encouraged many researchers to investigate the differential of that concept in the study of univalent functions. Many attempts have been applied this technique to the univalent and analytic functions to brought new results in this field. Generally, the authors [1], [2], [3], [4], and [5] with several other researchers are the persons who developed and contributed the study of the univalent function by using the tools of the concept of the subordination and superordination. The theory of the differential subordination in the complex analysis is given by the works of Miller and Mocanu [4] first. Later on, Bulboaca [5] investigated both differential subordination and superordination. Consequently, many authors continued the study on the theory of the differential subordination and superordination to determine the properties of the analytic and univalent functions. Thus, our focusing in this paper on the differential superordination in order to take a review on some of the recent developments in the differential superordination for analytic and univalent functions.

Let $\mathcal{H}$ be the class of the functions which are analytic in the open unit disk $\mathbb{U}=\{z:|z|<1\}$. For $\mathbb{N}=\{1,2,3,\ldots\}$ and $a\in\mathbb{C}$, let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}$ such that $\mathcal{H}(a,n)=\{f\in\mathcal{H}: f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\cdots\}$ and suppose that $\mathcal{A}=\mathcal{A}(1,1)$. Let f and g are analytic functions in $\mathbb{U}$, then the function f is said to be subordinate to g, or g is superordinate to f, if there exist a Schwarz function ω which analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$ for $z\in\mathbb{U}$. In such a case we write $f(z)\prec g(z)$ or $f\prec g$. More specifically, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence.

$$f(z)\prec g(z)\quad (z\in\mathbb{U})\Leftrightarrow f(0)=g(0),\, f(\mathbb{U})\subset g(\mathbb{U}).$$

Let $p,h\in\mathcal{H}$ and let $\phi(r,s,t;z):\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$. If p and $\phi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if p satisfies the following second order superordination

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z), \tag{1.1}$$

then p is the solution of the differential superordination (1,1). The analytic function g is called to be superordinate to f, if f is subordinate to g. The analytic function q is said to be a subordinant if $q\prec p$ for all p satisfying (1,1). The univalent subordinant $\tilde{q}$ is said to be the best subordinant if $q\prec\tilde{q}$ for all subordinants q of (1,1). In the next section, the essential definitions and the fundamental theorems which are concerning to the first, second and third order differential superordination are presented.

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2. DIFFERENTIAL SUPERORDINATION

In this section, we have investigated and summarized the differential superordination of some properties for analytic and univalent functions. The theory of first and second order differential superordination has been studied by several authors to solve some problems in this field (see [6] and [7]). For this purpose, the important results which are concerning to the differential superordination are presented in this paper.

The reference [8] studied the differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator. Then, some sufficient conditions for certain normalized analytic functions are presented such that the analytic functions $f(z)$ satisfying

$$q_1(z) \prec \left(\frac{H_m^l[\alpha_1] f(z)}{z} \right)^{\delta} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ. Furthermore, Juma *et al.* [9] provided some results for the second order differential superordination on analytic and multivalent functions in the open unit disk $\mathbb{U}$. Their results are obtained by investigating the appropriate classes of admissible functions which are given as follows.

Definition 3.1[9]: Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap H[0,p], zq'(z) \neq 0.$ The class of admissible function $\Psi'_n[\Omega,\mathrm{q}]$ consists of those function $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ hat satisfy the admissibility condition:

$$\phi(u,v,w,\xi) \in \Omega \text{ where } u = q(z),\ v = \frac{\frac{1}{k} zq'(z) + \frac{p(1-\lambda)}{\lambda} q(z)}{\frac{p}{\lambda}} \text{ and}$$

$$Re\left\{ \frac{p^2 w + 2p^2(1-\lambda)v - 3p^2(1-\lambda)^2 u}{\lambda p v - p\lambda(1-\lambda)u} \right\} \leq \frac{1}{k} Re\left\{ 1 + \frac{zq''(z)}{q'(z)} \right\}$$

where $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(q), \lambda > 0$ and $k \geq p.$

Theorem 3.1 [9]: Let $\phi \in \Psi_n[\Omega,\mathrm{q}]$. If $f \in \mathcal{A}(p), F_{\lambda,p}^m(f*g)(z) \in H_0$ and

$$\phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right), \text{ where } \lambda > 0, m \in N_0, z \in \mathbb{U}$$

implies that $q(z) \prec F_{\lambda,p}^m(f*g)(z), z \in \mathbb{U}.$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformed mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Psi'_n[h(\mathbb{U}),\mathrm{q}]$ is written as $\Psi'_n[h,\mathrm{q}]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2 [9]: Let $h(z)$ is analytic on $\mathbb{U}$ and $\phi \in \Psi'_n[h(\mathbb{U}),\mathrm{q}]$. If

$$f \in \mathcal{A}(p), F_{\lambda,p}^m(f*g)(z) \in H_0 \text{ and } \phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right), \text{ where } \lambda > 0, m \in N_0, z \in \mathbb{U},$$

implies that $q(z) \prec F_{\lambda,p}^m(f*g)(z), z \in \mathbb{U}.$

Theorems 3.1 and 3.2, can only be used to obtain subordinations of differential superordination of the form

$$\Omega \subset \phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right)$$

or

$$h(z) \prec \phi\left(F_{\lambda,p}^m(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right).$$

Tang *et al.* [10] investigated the problem of determining the properties of the functions $p(z)$ which are satisfying the following second order differential superordination:

$$\Omega \subset \left\{ \psi\left(p(z), p'(z), p''(z); z\right) : z \in \Delta \right\}.$$

The applications of the results to the second order differential superordination for analytic functions in Δ are also presented. For this aim, the class of admissible functions is given in the following definition.

Definition 3.1 [10]: Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[\Delta]$ with $zq'(z) \neq 0$. The class $\Phi'_\Delta[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, w, \xi) \in \Omega, \text{ whenever } u = q(z),\ v = \frac{q'(z)}{mq(z)},\ \left(q(z) \neq 0\right) \text{ and}$$

$$\Re\left(\frac{u(wv + v^2)}{q'(z)}\right) \leq \frac{1}{m^2} \Re\left(\frac{q''(z)}{q'(z)}\right) \left(z \in \Delta; \xi \in \partial\Delta; m > 0\right).$$

Theorem 3.2 [10]: Let $\phi \in \Phi'_\Delta[\Omega, q]$, $f(z) \neq 0$ and $f'(z) \neq 0$. If $f \in \mathcal{H}[\Delta]$, $\frac{f'(z)}{f(z)} \in \mathcal{Q}(\Delta)$ and

$$\phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z)f'''(z) - \left[f''(z)\right]^2\right]}{f'(z)\left[f(z)f''(z) - \left[f'(z)\right]^2\right]} - \frac{f'(z)}{f(z)}; z\right)$$

is univalent in Δ, then

$$\Omega \subset \left\{ \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z)f'''(z) - \left[f''(z)\right]^2\right]}{f'(z)\left[f(z)f''(z) - \left[f'(z)\right]^2\right]} - \frac{f'(z)}{f(z)}; z\right) : z \in \Delta \right\}, \text{ which}$$

implies that $q(z) \prec \frac{f'(z)}{f(z)} \quad (z \in \Delta)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\Delta)$ for some conformed mapping $h(z)$ of Δ onto Ω, then the class $\Phi'_\Delta\left[h(\Delta), q\right]$ is written simply as $\Phi'_\Delta[h, q]$. The following result is an immediate consequence of Theorem 3.2.

Theorem 3.3 [10]: Let $q \in \mathcal{H}[\Delta]$. Also let the function $h(z)$ be analytic in Δ and $\phi \in \Phi'_\Delta[\Omega, q]$.

If $f \in \mathcal{H}[\Delta]$ with $f(z) \neq 0$ and $f'(z) \neq 0$, $\frac{f'(z)}{f(z)} \in \mathcal{Q}(\Delta)$ and

$$\phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z)f'''(z) - \left[f''(z)\right]^2\right]}{f'(z)\left[f(z)f''(z) - \left[f'(z)\right]^2\right]} - \frac{f'(z)}{f(z)}; z\right)$$

is univalent in Δ, then

$$h(z) \prec \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z)f'''(z) - \left[f''(z)\right]^2\right]}{f'(z)\left[f(z)f''(z) - \left[f'(z)\right]^2\right]} - \frac{f'(z)}{f(z)}; z\right) \quad (z \in \Delta)$$

implies that $q(z) \prec \frac{f'(z)}{f(z)} \ (z \in \Delta)$.

Theorems 3.2 and 3.3 can only be used to obtain subordinations involving the differential superordination of the form

$$\Omega \subset \left\{ \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z) f'''(z) - \left[f''(z) \right]^2 \right]}{f'(z)\left[f(z) f''(z) - \left[f'(z) \right]^2 \right]} - \frac{f'(z)}{f(z)}; z \right) : z \in \Delta \right\} \text{ or}$$

$$h(z) \prec \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z) f'''(z) - \left[f''(z) \right]^2 \right]}{f'(z)\left[f(z) f''(z) - \left[f'(z) \right]^2 \right]} - \frac{f'(z)}{f(z)}; z \right).$$

The following theorem proves the existence of the best subordinant of

$$h(z) \prec \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z) f'''(z) - \left[f''(z) \right]^2 \right]}{f'(z)\left[f(z) f''(z) - \left[f'(z) \right]^2 \right]} - \frac{f'(z)}{f(z)}; z \right) \text{ for a suitably chosen } \phi.$$

Theorem 3.4 [10]: Let the function $h(z)$ be analytic in $\Delta,$ and let $\phi : \mathbb{C}^3 \times \overline{\Delta} \to \mathbb{C}.$ Suppose that the following differential equation:

$$\phi\left(q(z), \frac{q'(z)}{q(z)} - \frac{q''(z)}{q'(z)} - \frac{q'(z)}{q(z)}; z \right) = h(z)$$

has a solution $q \in \mathcal{Q}(\Delta).$ If $\phi \in \Phi'_{\Delta}[h, q], f \in \mathcal{H}[\Delta]$ with $f(z) \neq 0$ and $f'(z) \neq 0,$ $\frac{f'(z)}{f(z)} \in \mathcal{Q}(\Delta)$ with

$$\phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z) f'''(z) - \left[f''(z) \right]^2 \right]}{f'(z)\left[f(z) f''(z) - \left[f'(z) \right]^2 \right]} - \frac{f'(z)}{f(z)}; z \right)$$

is univalent in $\Delta,$ then

$$h(z) \prec \phi\left(\frac{f'(z)}{f(z)}, \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}, \frac{f(z)\left[f'(z) f'''(z) - \left[f''(z) \right]^2 \right]}{f'(z)\left[f(z) f''(z) - \left[f'(z) \right]^2 \right]} - \frac{f'(z)}{f(z)}; z \right) \quad (z \in \Delta)$$

implies that $q(z) \prec \frac{f'(z)}{f(z)} \; (z \in \Delta),$ and $q(z)$ is the best subordinant.

A few articles have been discussed the third order differential subordination and superordination such as [11] and [12]. Again, Tang *et al.* [13] generalized second order differential superordination to introduced the concept of the third order differential superordination which given as follows.

Definition 22 [13]: Let Ω be a set in $\mathbb{C}$ and $q \in \mathfrak{P}$ with $q'(z) \neq 0.$ The class of admissible functions $\Phi'_H[\Omega, q]$ consist of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition: $\phi(a, b, c, d, \xi) \in \Omega$

where $a = q(z), \; b = \dfrac{zq'(z) + m\beta_1 q(z)}{m\beta_1},$

$$\Re\left(\frac{(\beta_1 + 1)(c - a)}{b - a} - (2\beta_1 + 1) \right) \leq \frac{1}{m} \Re\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$$\Re\left(\frac{(\beta_1 + 1)(\beta_1 + 2)(d - 3c + 3b - a)}{b - a} \right) \leq \frac{1}{m^2} \Re\left\{ \frac{z^2 q'''(z)}{q'(z)} \right\}$$

where $z \in \mathbb{U}, \beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \xi \in \partial\mathbb{U}$ and $m \in \mathbb{N} \setminus \{1\}.$

Theorem 23 [13]: Let $\phi \in \Phi'_H[\Omega, \mathrm{q}]$. If the functions $f \in \sum_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \in \mathfrak{Q}_1$ satisfy the following conditions: $\Re\left(\frac{z q''(z)}{q'(z)}\right) \geq 0,\ \left|\frac{z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z)}{q'(z)}\right| \leq m,$

$$\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\begin{array}{l} z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), \\ z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z \end{array}\right) : z \in \mathbb{U}\right\}$$

implies that $q(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, then the class $\Phi'_H[h(\mathbb{U}), \mathrm{q}]$ can be written as $\Phi'_H[h, \mathrm{q}]$. As a consequence, the following theorem is an immediate result of theorem 23.

Theorem 24 [13]: Let $\phi \in \Phi'_H[\Omega, \mathrm{q}]$. Also let the function h be analytic in $\mathbb{U}$. If the functions $f \in \sum_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \in \mathfrak{Q}_1$ satisfy the condition $\Re\left(\frac{z q''(z)}{q'(z)}\right) \geq 0,\ \left|\frac{z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z)}{q'(z)}\right| \leq m,$ and

$$\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right)$$

implies that $q(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z)$.

Theorems 23 and 24 can be used to obtain the subordinations involving the third order differential superordination of the following forms.

$$\Omega \subset \left\{\phi\left(\begin{array}{l} z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), \\ z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z \end{array}\right) : z \in \mathbb{U}\right\}$$

or

$$h(z) \prec \phi\left(\begin{array}{l} z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), \\ z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z \end{array}\right).$$

For a suitable chosen ϕ, the following theorem proves the existence of the best subordinant of the form

$$h(z) \prec \phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right).$$

Theorem 25 [13]: Let the function h be analytic in $\mathbb{U}$, and let $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ and ψ be given by

$$\psi(r,s,t,u;z) = \phi(a,b,c,d;z) =$$

$$\phi\left(r, \frac{s+\beta_1 r}{\beta_1}, \frac{t+2(\beta_1+1)s+\beta_1(\beta_1+1)r}{\beta_1(\beta_1+1)}, \frac{u+3(\beta_1+2)t+3(\beta_1+1)(\beta_1+2)s+\beta_1(\beta_1+1)(\beta_1+2)r}{\beta_1(\beta_1+1)(\beta_1+2)}; z\right).$$

Suppose that the differential equation $\psi\left(q(z), zq'(z), z^2 q''(z) z^3 q'''(z); z\right) = h(z)$ has a solution $q(z) \in \mathfrak{Q}_1$. If the functions $f \in \sum_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \in \mathfrak{Q}_1$ satisfy the condition

$$\Re\left(\frac{zq''(z)}{q'(z)}\right)\geq 0,\ \left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z)}{q'(z)}\right|\leq m,\ \text{and}$$

$$\phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2)f(z);z\right).$$

is univalent in $\mathbb{U}$, then

$$h(z)\prec\phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1)f(z),z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2)f(z);z\right),$$

implies that $q(z)\prec z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z)$ and q is the best subordinant.

Consequently, some other studies investigated the third order differential superordination based on the analytic functions by involving some specific functions. [14] obtained some third order differential superordination by involving the generalized Bessel functions, and with the functions associated with the operator B_k^c defined as follows.

$B_k^c f(z):\varphi_{k,c}(z)*f(z)=z+\sum_{n=1}^{\infty}\frac{(-c)^n a_{n+1}z^{n+1}}{4^n (k)_n n!}$, in terms of the Taylor-Maclaurin coefficients a_{n+1} involved in

$$f(z)=z+\sum_{n=1}^{\infty}a_{n+1}z^{n+1}\quad(z\in\mathbb{U}).$$

For this goal, the following definition gives the class of admissible functions.

Definition 6 [14]: let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{H}_0$ with $q'(z)\neq 0.$ The class of admissible functions $\Phi'_B[\Omega,\mathrm{q}]$ consists of those functions $\phi:\mathbb{C}^4\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the admissibility condition $\phi(\alpha,\beta,\gamma,\delta;\xi)\in\Omega$

whenever $\alpha=q(z),\ \beta=\dfrac{zq'(z)+m(k-1)q(z)}{mk},$

$$\Re\left(\frac{k(k-1)\gamma-(k-1)(k-2)\alpha}{k\beta-(k-1)\alpha}-(2k-3)\right)\leq\frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)}+1\right)\ \text{and}$$

$$\Re\left(\frac{k(k-1)(1-k)\alpha+3k\beta+(1-3k)\gamma+(k-2)\delta}{\alpha+k(\beta-\alpha)}\right)\leq\frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in\mathbb{U},k\in\mathbb{C}\setminus\{0,1,2\},\xi\in\partial\mathbb{U}$ and $m\geq 2.$

Theorem 5 [14]: Let $\phi\in\Phi'_B[\Omega,\mathrm{q}].$ If the functions $f\in\mathrm{A},B_{k+1}^c f(z)\in\mathcal{Q}_0$ and $q\in\mathcal{H}_0$ with $q'(z)\neq 0$ satisfying the following conditions:

$\Re\left(\dfrac{zq''(z)}{q''(z)}\right)\geq 0,\ \left|\dfrac{B_k^c f(z)}{q''(z)}\right|\leq m,$ and $\phi\left(B_{k+1}^c f(z),B_k^c f(z),B_{k-1}^c f(z),B_{k-2}^c f(z);z\right)$ is univalent in $\mathbb{U},$

then $\Omega\subseteq\left\{\phi\left(B_{k+1}^c f(z),B_k^c f(z),B_{k-1}^c f(z),B_{k-2}^c f(z);z\right):z\in\mathbb{U}\right\}$

implies that $q(z)\prec B_{k+1}^c f(z)\ \ (z\in\mathbb{U}).$

If $\Omega\neq\mathbb{C}$ is a simply connected domain and $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega,$ then the class $\Phi'_B[h(\mathbb{U}),\mathrm{q}]$ is written as $\Phi'_B[h,\mathrm{q}]$. The following result is an immediate consequence of Theorem 5.

Theorem 6 [14]: Let $\phi\in\Phi'_B[\Omega,\mathrm{q}]$ and h be analytic in $\mathbb{U}.$ If the functions $f\in\mathrm{A},B_{k+1}^c f(z)\in\mathcal{Q}_0$ and $q\in\mathcal{H}_0$ with $q'(z)\neq 0$ satisfying the following condition

$$\Re\left(\frac{zq''(z)}{q'(z)}\right)\geq 0,\ \left|\frac{B_k^c f(z)}{q'(z)}\right|\leq m,\ \text{and}$$

$\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right)$ is univalent in $\mathbb{U}$, then

$h(z)\prec\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right)$ implies that $q(z)\prec B_{k+1}^{c}f(z)\quad(z\in\mathbb{U})$.

Theorems 5 and 6 can only be used to obtain subordinations of third order differential superordination of the following forms

$$\Omega\subseteq\left\{\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right):z\in\mathbb{U}\right\}$$

or

$$h(z)\prec\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right).$$

The following theorem proves the existence of the best subordinant of

$h(z)\prec\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right)$ for a suitable ϕ.

Theorem 7 [14]: Let h be analytic in $\mathbb{U}$, and $\phi:\mathbb{C}^{4}\times\overline{\mathbb{U}}\to\mathbb{C}$ with ψ be given by

$$\psi(r,s,t,u;z)=\phi(\alpha,\beta,\gamma,\delta;z)=\phi\left(\begin{array}{l} r,\dfrac{s+(k-1)r}{k},\dfrac{t+2(k-1)s+(k-1)(k-2)r}{k(k-1)},\\ \dfrac{u+3(k-1)t+3(k-1)(k-2)s+(k-1)(k-2)(k-3)r}{k(k-1)(k-2)};z\end{array}\right).$$

Suppose that the differential equation $\psi\left(q(z),zq'(z),z^{2}q''(z)z^{3}q'''(z);z\right)=h(z)$ has a solution $q(z)\in\mathcal{Q}_{0}$. If the functions $f\in\mathrm{A},B_{k+1}^{c}f(z)\in\mathcal{Q}_{0}$ and $q\in\mathcal{H}_{0}$ with $q'(z)\neq0$ satisfying the condition

$\Re\left(\dfrac{zq''(z)}{q'(z)}\right)\geq0,\ \left|\dfrac{B_{k}^{c}f(z)}{q'(z)}\right|\leq m,$ and $\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right)$ is univalent in $\mathbb{U}$,

then $h(z)\prec\phi\left(B_{k+1}^{c}f(z),B_{k}^{c}f(z),B_{k-1}^{c}f(z),B_{k-2}^{c}f(z);z\right)$ implies that $q(z)\prec B_{k+1}^{c}f(z)\quad(z\in\mathbb{U})$ and $q(z)$ is the best subordinant.

Furthermore, [15] utilized the methods of the third order differential superordination results of [12] and [13], respectively. They investigated some applications of the third order differential superordination of analytic functions associated with the new operator. Then, suitable classes of admissible functions are considered as follows.

Definition 4.1 [15]: Let Ω be a set in $\mathbb{C}$, $c_{\alpha},c_{\alpha+1},c_{\alpha+2},\in\mathbb{C}\setminus\{0\}$ and $q\in\mathcal{H}_{0}$ with $q'(z)\neq0$. The class of admissible functions $\Phi_{T}'[\Omega,\mathrm{q}]$ consists of those functions $\phi:\mathbb{C}^{4}\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the following admissibility condition: $\phi(v,w,x,y;\xi)\in\Omega$ where $v=q(z)$,

$$w=\frac{zq'(z)+m(c_{\alpha}-1)q(z)}{mc_{\alpha}},$$

$$\Re\left(\frac{c_{\alpha}c_{\alpha+1}x-(c_{\alpha}-1)(c_{\alpha+1}-1)v}{c_{\alpha}w-(c_{\alpha}-1)v}-(c_{\alpha}+c_{\alpha+1}-2)\right)\leq\frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)}+1\right)\text{ and}$$

$$\Re\left[\frac{(c_{\alpha}c_{\alpha+1}c_{\alpha+2})y-(c_{\alpha}-1)(c_{\alpha+1}-1)(c_{\alpha+2}-1)v}{c_{\alpha}w-(c_{\alpha}-1)v}-(c_{\alpha}+c_{\alpha+1}+c_{\alpha+2})\right.$$

$$\left.\left[\frac{c_{\alpha}c_{\alpha+1}x-(c_{\alpha}-1)(c_{\alpha+1}-1)v}{c_{\alpha}w-(c_{\alpha}-1)v}-(c_{\alpha}+c_{\alpha+1}-1)\right]-\left(c_{\alpha+2}(c_{\alpha}+c_{\alpha+1}-1)+(c_{\alpha}-1)(c_{\alpha+1}-1)\right)\right]\leq$$

$$\frac{1}{m^{2}}\Re\left(\frac{z^{2}q'''(z)}{q'(z)}\right),\text{ where } z\in\mathcal{U},\xi\in\partial\mathcal{U}\setminus E(q)\text{ and } m\geq2.$$

Theorem 4.2 [15]: Let $\phi \in \Phi'_T[\Omega, q]$. If the function $f \in A, T_\alpha f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the following conditions: $\Re\left(\frac{z q''(z)}{q''(z)}\right) \geq 0, \left|\frac{T_{\alpha+1} f(z)}{q'(z)}\right| \leq m,$ and

$\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)$ is univalent in u, then

$$\Omega \subseteq \left(\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z \in \mathcal{U})\right)$$

implies that $q(z) \prec T_\alpha f(z) \ (z \in \mathcal{U})$.

Theorem 4.3 [15]: Let $\phi \in \Phi'_T[\Omega, q]$ and the function h be analytic in $\mathcal{U}$. If the function $f \in A, T_\alpha f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the following conditions

$$\Omega \subseteq \left(\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z \in \mathcal{U})\right)$$

and

$\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)$, is univalent in $\mathcal{U}$, then

$$h(z) \prec \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)$$

implies that $q(z) \prec T_\alpha f(z) \ (z \in \mathcal{U})$.

Theorem 4.2 and 4.3 can only be used to obtain subordinations of the third order differential superordination of the forms

$$h(z) \prec \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)$$

or

$$\left(\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z)\right); z\right) = h(z).$$

The following theorem proves the existence of the best subordinant of

$$\left(\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z)\right); z\right) = h(z)$$

for a suitable chosen ϕ.

Theorem 4.4 [15]: Let the function h be analytic in $\mathcal{U}$ and let $\phi : \mathbb{C}^4 \times \mathcal{U} \to \mathbb{C}$ and ψ be given by

$$\psi\left(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z\right) = \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z).$$

Suppose that the differential equation $\left(\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z)\right); z\right) = h(z)$ has a solution $q(z) \in \mathcal{Q}_0$. If the function $f \in A, T_\alpha f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the following conditions

$$\Omega \subseteq \left(\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z \in \mathcal{U})\right)$$

and

$$\phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z),$$

is univalent in $\mathcal{U}$, then

$$h(z) \prec \phi(T_\alpha f(z), T_{\alpha+1} f(z), T_{\alpha+2} f(z), T_{\alpha+3} f(z); z)$$

implies that $q(z) \prec T_\alpha f(z) \ (z \in \mathcal{U})$ and q is the best subordinant.

3. CONCLUSION

As a conclusion, the differential superordination in the theory of analytic functions are investigated. Some of the recent results concerning on the first, second and third order differential superordination based on the analytic function are summarized.

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