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DECOMPOSITION AND GENERALIZED INVERSE FOR QUASI-ROW (COLUMN) SYMMETRIC MATRIX

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ABSTRACT

In this paper the quasi-row(column) symmetric matrix is considered, its full rank decomposition and singular decomposition are studied, and the generalized inverse for any decomposition is obtained, the relationship between mother matrix and quasi-row(column) symmetric matrix is derived. The results show that the method can not only reduce the calculated amount and memory space, but also can not reduce the numerical accuracy; so extend the results of the related references and spread its application scope.

Keywords: *quasi-row (column) symmetric matrix; full rank decomposition; singular decomposition; Moore-Penrose inverse*

1. INTRODUCTION

Singular value decomposition, full rank decomposition is the most commonly used method in matrix decomposition, and it plays important roles in finding the Moore-Penrose inverse of the matrix, therefore, the study of symmetric matrix is of great importance. The literature [2] studied the polar decomposition and perturbation bounds of the quasi-row (column) symmetric matrix with the matrix Q as the permutation matrix, The literature [3] studied the singular value decomposition of the extension matrix with $Q_1 = Q_2 = \cdots = Q_{k-1} = Q$ as the permutation matrix, The literature [4] studied the Moore-Penrose inverse of generalized row (column) symmetric matrix, such as $\begin{pmatrix} A \\ QA \end{pmatrix}, \begin{pmatrix} A \\ AQ \end{pmatrix}$. In this paper, based on the literature [3-4], the quasi-row(column) symmetric matrix is studied again, and the conclusion in the literature [2] is generalized, the permutation matrix mentioned in this paper is a matrix obtained by the unit matrix being exchanged several times rows (or columns).

2. QUASI-ROW (COLUMN) SYMMETRIC MATRIX

Definition 1[2]: Let $A \in \mathbb{C}^{m\times n}, Q_1, Q_2, \cdots, Q_{k-1}$ is permutation matrix, then

$$R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = \begin{pmatrix} A \\ A_1 \\ \vdots \\ A_{k-1} \end{pmatrix} = \begin{pmatrix} A \\ Q_1 A \\ \vdots \\ Q_{k-1} A \end{pmatrix}$$

is called the k-th quasi row symmetric matrix of A, and A is its mother matrix.

Definition 2[2: Let $A \in \mathbb{C}^{m\times n}, Q_1, Q_2, \cdots, Q_{k-1}$ is permutation matrix, then

$$C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = \begin{pmatrix} A & A_1 & \cdots & A_{k-1} \end{pmatrix} = \begin{pmatrix} A & AQ_1 & \cdots & AQ_{k-1} \end{pmatrix}$$

is called the k-th quasi column symmetric matrix of A, and A is its mother matrix.
When $Q_1 = Q_2 = \cdots = Q_{k-1} = Q$, we denote

$$R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = R_k\left(A; Q\right), C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = C_k\left(A; Q\right).$$

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We can easily obtain that $R_k\left(A;Q\right), C_k\left(A;Q\right)$ is the "continuation matrix" in the literature [3].

3. THE FULL RANK DECOMPOSITION AND GENERALIZED INVERSE FOR QUASI-ROW (COLUMN) SYMMETRIC MATRIX

Theorem 1: Let the full rank decomposition of $A \in \mathbb{C}^{m\times n}$ is $A = FG$,
where $F \in \mathbb{C}^{m\times r}$ is column full rank decomposition, $G \in \mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA = rankF = rankG = r\left(r>0\right)$, then

$$R\left(A;Q_1,Q_2,\cdots,Q_{k-1}\right) = \begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix} G$$

is the full rank decomposition of quasi row symmetric matrix.

Proof: According to the conditions, we have $\begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix} \in \mathbb{C}^{km\times r}$, by the rank preserving in the literature [2],

$rank\begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix} = rankF = r$, so, $\begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix}$ is column full rank matrix, from $G \in \mathbb{C}^{r\times n}$, $rankG = r$ is row full rank matrix, such that

$$\begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix} G = \begin{pmatrix} FG \\ Q_1FG \\ \vdots \\ Q_{k-1}FG \end{pmatrix} = \begin{pmatrix} A \\ Q_1A \\ \vdots \\ Q_{k-1}A \end{pmatrix} = R\left(A;Q_1,Q_2,\cdots,Q_{k-1}\right).$$

Theorem 2: Let the full rank decomposition of $A \in \mathbb{C}^{m\times n}$ is $A = FG$, where $F \in \mathbb{C}^{m\times r}$ is column full rank decomposition, $G \in \mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA = rankF = rankG = r\left(r>0\right)$, then

$$C\left(A;Q_1,Q_2,\cdots,Q_{k-1}\right) = F\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}$$

is the full rank decomposition of quasi column symmetric matrix.

Proof: According to the conditions, we have $\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix} \in \mathbb{C}^{r\times kn}$, by the rank preserving in the literature [2], $rank\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix} = rankG = r$, so, $\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}$ is row full rank matrix, from $F \in \mathbb{C}^{m\times r}$, $rankF = r$ is column full rank matrix, such that

$$\begin{aligned} F\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix} &= \begin{pmatrix} FG & FGQ_1 & \cdots & FGQ_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} A & AQ_1 & \cdots & AQ_{k-1} \end{pmatrix} = C\left(A;Q_1,Q_2,\cdots,Q_{k-1}\right). \end{aligned}$$

Lemma 1[1]: Let the full rank decomposition of $A \in \mathbb{C}^{m\times n}$ is $A = FG$, then

$$A^+ = G^+F^+ = G^H\left(GG^H\right)^{-1}\left(F^HF\right)^{-1}F^H$$

Theorem 3: Let the full rank decomposition of $A \in \mathbb{C}^{m\times n}$ is $A = FG$, where $F \in \mathbb{C}^{m\times r}$ is column full rank decomposition, $G \in \mathbb{C}^{r\times n}$ is row full rank matrix, and $rankA = rankF = rankG = r\left(r > 0\right)$, then

(1) The Moore-Penrose inverse of the quasi row symmetric matrix $R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$ is:

$$\left[R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ = \frac{1}{k}\begin{pmatrix} A^+ & A^+Q_1^H & \cdots & A^+Q_{k-1}^H \end{pmatrix},$$

(2) The Moore-Penrose inverse of the quasi column symmetric matrix $C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$ is:

$$\left[C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ = \frac{1}{k}\begin{pmatrix} A^+ \\ Q_1^H A^+ \\ \vdots \\ Q_{k-1}^H A^+ \end{pmatrix}.$$

Proof: Let the full rank decomposition of $A \in \mathbb{C}^{m\times n}$ is $A = FG$, by theorem 1, we have

$$R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = \begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix} G$$

is the full rank decomposition of $R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$,
and
$C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = F\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}$ is the full rank decomposition of $C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$,
by lemma 1,we have $A^+ = G^+F^+ = G^H\left(GG^H\right)^{-1}\left(F^HF\right)^{-1}F^H$,therefore:

$$\left[R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ = G^H\left(GG^H\right)^{-1}\left[\begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix}^H \begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix}\right]^{-1} \begin{pmatrix} F \\ Q_1F \\ \vdots \\ Q_{k-1}F \end{pmatrix}^H$$

$$= G^H\left(GG^H\right)^{-1}\left[kF^HF\right]^{-1}\begin{pmatrix} F^H & F^HQ_1^H & \cdots & F^HQ_{k-1}{}^H \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} A^+ & A^+Q_1^H & \cdots & A^+Q_{k-1}^H \end{pmatrix},$$

$$\left[C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+$$

$$= \begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}^H \left[\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}\begin{pmatrix} G & GQ_1 & \cdots & GQ_{k-1} \end{pmatrix}^H\right]^{-1}\left(F^HF\right)^{-1}F^H$$

$$= \begin{pmatrix} G^H & Q_1{}^HG^H & \cdots & Q_{k-1}{}^HG^H \end{pmatrix}^T\left[kGG^H\right]^{-1}\left(F^HF\right)^{-1}F^H$$

$$= \frac{1}{k}\begin{pmatrix} A^+ \\ Q_1^H A^+ \\ \vdots \\ Q_{k-1}^H A^+ \end{pmatrix}.$$

4. THE SINGULAR DECOMPOSITION AND GENERALIZED INVERSE FOR QUASI-ROW (COLUMN) SYMMETRIC MATRIX

Theorem 4: Let the singular decomposition of $A \in \mathbb{C}^{m\times n}\left(rankA = r > 0\right)$ is $A = UDV^H$, where U is a m-order unitary matrix, V is an n-order unitary matrix, $D=\begin{pmatrix}\Sigma & O\\ O & O\end{pmatrix}$, $\Sigma = diag(\sigma_1,\sigma_2,\cdots,\sigma_r)$ and the $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ is the positive singular value of matrix A. Then there is singular value decomposition for the quasi row symmetric matrix $R(A;Q_1,Q_2,\cdots,Q_{k-1}) \in \mathbb{C}^{km\times n}$:

$$R(A;Q_1,Q_2,\cdots,Q_{k-1}) = XTV^H$$

if: (1) $X=(X_1,X_2)$ where $X_1=R\left(\frac{1}{\sqrt{k}}U;Q_1,Q_2,\cdots,Q_{k-1}\right)$, $X_2 \in \mathbb{C}^{m+1\times km}$, $X_2^H X_1 = 0$,

such that $X=(X_1,X_2)$ is a unitary matrix;

(2) $T=\begin{pmatrix}\Delta & O\\ O & O\end{pmatrix}$, $\Delta = diag\left(\sqrt{k}\sigma_1,\sqrt{k}\sigma_2,\cdots,\sqrt{k}\sigma_r\right)$.

Proof: 1) Form

$$R(A;Q_1,Q_2,\cdots,Q_{k-1})^H R(A;Q_1,Q_2,\cdots,Q_{k-1}) = kA^H A = kVD^H U^H UDV^H = VkD^2V^H,$$

So, $\sqrt{k}\sigma_1,\sqrt{k}\sigma_2,\cdots,\sqrt{k}\sigma_r$ is the positive singular value of $R(A;Q_1,Q_2,\cdots,Q_{k-1})^H R(A;Q_1,Q_2,\cdots,Q_{k-1})$, according to the definition of the singular value, we know that $\sqrt{k}\sigma_1,\sqrt{k}\sigma_2,\cdots,\sqrt{k}\sigma_r$ must be singular value of $R(A;Q_1,Q_2,\cdots,Q_{k-1})$.

2) Suppose $X_1=R\left(\frac{1}{\sqrt{k}}U;Q_1,Q_2,\cdots,Q_{k-1}\right)$ then

$$X_1^H X_1=\begin{pmatrix}\frac{1}{\sqrt{k}}U^H & \frac{1}{\sqrt{k}}U^H Q_1^H & \cdots & \frac{1}{\sqrt{k}}U^H Q_{k-1}^H\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{k}}U\\ Q_1\frac{1}{\sqrt{k}}U\\ \vdots\\ Q_{k-1}\frac{1}{\sqrt{k}}U\end{pmatrix}=I.$$

Therefore, there is another arbitrary matrix X_2 such that $X=(X_1,X_2)$ is a unitary matrix and $X_2^H X_1 = 0$.

3) Since $X^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V = \begin{pmatrix}X_1^H\\ X_2^H\end{pmatrix}R(A;Q_1,Q_2,\cdots,Q_{k-1})V = \begin{pmatrix}X_1^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V\\ X_2^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V\end{pmatrix}$

and,

$$X_1^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V = \begin{pmatrix}\frac{1}{\sqrt{k}}U^H & \frac{1}{\sqrt{k}}U^H Q_1^H & \cdots & \frac{1}{\sqrt{k}}U^H Q_{k-1}^H\end{pmatrix}\begin{pmatrix}A\\ Q_1A\\ \vdots\\ Q_{k-1}A\end{pmatrix}V$$

$$=\sqrt{k}U^H AV = kD^2 = T.$$

but,

$$X_2^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V = X_2^H R(AV;Q_1,Q_2,\cdots,Q_{k-1}) = X_2^H R(UD;Q_1,Q_2,\cdots,Q_{k-1})$$

$$= X_2^H R\left(\frac{1}{\sqrt{k}}U;Q_1,Q_2,\cdots,Q_{k-1}\right)\sqrt{k}D = X_2^H X_1\sqrt{k}D = 0.$$

Thus, we have $X^H R(A;Q_1,Q_2,\cdots,Q_{k-1})V = T$.

Theorem 5: Let the singular decomposition of $A \in \mathbb{C}^{m\times n}\left(rankA = r > 0\right)$ is $A = UDV^H$, where U is a m-order unitary matrix, V is an n-order unitary matrix, $D = \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix}$, $\Sigma = diag\left(\sigma_1, \sigma_2, \cdots, \sigma_r\right)$ and the $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ is the positive singular value of matrix A. Then there is singular value decomposition for the quasi column symmetric matrix $C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) \in \mathbb{C}^{m\times kn}$:

$$C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = UTY^H$$

if: (1) $Y = \left(Y_1, Y_2\right)$ where $Y_1 = C\left(\frac{1}{\sqrt{k}}V^H; Q_1^H, Q_2^H, \cdots, Q_{k-1}^H\right)$, $Y_2 \in \mathbb{C}^{n+1\times kn}$, $Y_2^H Y_1 = 0$,

such that $Y = \left(Y_1, Y_2\right)$ is a unitary matrix;

(2) $T = \begin{pmatrix} \Delta & O \\ O & O \end{pmatrix}$, $\Delta = diag\left(\sqrt{k}\sigma_1, \sqrt{k}\sigma_2, \cdots, \sqrt{k}\sigma_r\right)$.

The proof is similar to Theorem 4, and it is no longer proved here.

Lemma 2[1]: Let the singular decomposition of $A \in \mathbb{C}^{m\times n}\left(rankA = r > 0\right)$ is $A = UDV^H$, where U is a m-order unitary matrix, V is an n-order unitary matrix, $D = \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix}$, $\Sigma = diag\left(\sigma_1, \sigma_2, \cdots, \sigma_r\right)$ and the $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ is the positive singular value of matrix A. Then

$$A^+ = VD^{-1}U^H$$

is the Moore-Penrose inverse of matrix A.

Theorem 6: Let the singular decomposition of $A \in \mathbb{C}^{m\times n}\left(rankA = r > 0\right)$ is $A = UDV^H$, then:

(1) The Moore-Penrose inverse of the quasi row symmetric matrix $R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$ is:

$$\left[R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ = \frac{1}{k}\begin{pmatrix} A^+ & A^+Q_1^H & \cdots & A^+Q_{k-1}^H \end{pmatrix};$$

(2) The Moore-Penrose inverse of the quasi column symmetric matrix $C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$ is:

$$\left[C\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ = \frac{1}{k}\begin{pmatrix} A^+ \\ Q_1^H A^+ \\ \vdots \\ Q_{k-1}^H A^+ \end{pmatrix}.$$

Proof: By theorem 4, $R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right) = XTV^H$ is the singular value decomposition for the quasi row symmetric matrix $R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)$, thus, by lemma 2,

$$\begin{aligned}\left[R\left(A; Q_1, Q_2, \cdots, Q_{k-1}\right)\right]^+ &= VT^{-1}X^H = \frac{1}{\sqrt{k}}VD^{-1}X_1^H \\ &= \frac{1}{k}\begin{pmatrix} VD^{-1}U^H & VD^{-1}U^H Q_1^H & \cdots & VD^{-1}U^H Q_{k-1}^H \end{pmatrix} \\ &= \frac{1}{k}\begin{pmatrix} A^+ & A^+Q_1^H & \cdots & A^+Q_{k-1}^H \end{pmatrix}.\end{aligned}$$

The proof of conclusion (2) is similar to conclusion (1), we no longer prove it again.

5. CONCLUSIONS

To sum up, for the full rank decomposition and singular value decomposition of the quasi row (column) symmetric matrices, the full rank decomposition and singular value decomposition of the mother matrix A can be carried out first, and then its full rank decomposition and singular value decomposition can be easily calculated by using the theorems in this paper. For the Moore-Penrose inverse of quasi row (column) symmetric matrix, we found that whether it is full rank decomposition or singular value decomposition to solve Moore-Penrose inverse, the results are the same, it shows that for solving the Moore-Penrose inverse of the quasi row (column) symmetric matrix, the Moore-Penrose inverse of the mother matrix A can be first calculated and then using the theorems in this paper. Therefore, the decomposition and generalized inverse of quasi row (column) symmetric matrix can be calculated by the decomposition and generalized inverse of the mother matrix A, which not only reduces the computation but also does not lose accuracy.

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