International Research Journal of Pure Algebra-9(1), 2019, 1-10
Available online through www.rjpa.info ISSN 2248–9037

REALIZATIONS OF THREE-DIMENSIONAL GROUP ACTIONS

SIMON DAVIS
Research Foundation of Southern California
8861 Villa La Jolla Drive #13595
La Jolla, CA 92039.

(Received On: 24-10-18; Accepted On: 30-01-19)

ABSTRACT

The minimum dimensions of the real representations of the three-dimensional Lie algebras defining the maximal transitive groups on the basic three-geometries are given. The vector fields on Nil geometry are used to define a differential operator, and the kernel is found. The dynamics of Ricci solitons on this manifold and the embedding in four-dimensions are described.

Keywords: *three dimensions, vector fields, Ricci solitons, embedding.*

MSC: *51M15, 53C10.*

1. INTRODUCTION

The geometrization of three-dimensional manifolds has been developed from the fundamental theorem on the enumeration of geometries with compact models and point stabilizer groups [1]. The basic three-geometries have characteristic isometry groups, and their realizations will be considered. The Bianchi Lie algebras have nontrivial commutation relations and represent all except one of the basic three-geometries. Special consideration will be given to the algebras for the Nil and Sol geometries.

The Lie algebras are demonstrated to have a real representation in three dimensions. The solutions to the equations derived from the commutators are found for the Nil and Sol algebras. Then the vector fields in $\mathbf{R}^3$ that are tangent to the manifold are used to define a differential operator. The kernel of this operator is determined and the relation to other operators is established through the characteristics of the algebra. These results are established for the Nil geometry.

The Euclidean and Lorentz solitons are described. The embedding of these solitons in four dimensions is then considered. It is found that the condition of fixed volume with Euclidean signature does not allow such an embedding. If the manifold is noncompact such as $\mathbf{H}^4$, the embedding is not allowed if the volume of the soliton increases too rapidly with time. The necessity of introducing a Lorentz signature then would follow.

2. COMMUTATION RELATIONS OF THREE-DIMENSIONAL LIE ALGEBRAS

The commutators of the Nil algebra are

$$[e_1, e_2] = e_3$$
$$[e_1, e_3] = 0 \qquad (2.1)$$
$$[e_2, e_3] = 0$$

Defining

$$e_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$e_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad (2.2)$$

Corresponding Author: Simon Davis
Research Foundation of Southern California
8861 Villa La Jolla Drive #13595
La Jolla, CA 92039.

$$e_3 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

the equations

$$\begin{aligned}
&a_{12}b_{21} - a_{21}b_{12} = c_{11} \\
&a_{11}b_{12} + a_{12}b_{22} - a_{12}b_{11} - a_{22}b_{12} = c_{12} \\
&a_{21}b_{11} + a_{22}b_{21} - b_{21}a_{11} - b_{22}a_{12} = c_{21} \\
&a_{21}b_{12} - b_{21}a_{12} = c_{22} \\
&a_{12}c_{21} - c_{12}a_{21} = 0 \\
&a_{11}c_{12} + a_{12}c_{22} - a_{12}c_{11} - a_{22}c_{12} = 0 \\
&a_{21}c_{11} + a_{22}c_{21} - a_{11}c_{21} - a_{21}c_{22} = 0 \\
&a_{21}c_{12} + a_{12}c_{22} - a_{12}c_{21} - a_{22}c_{12} = 0 \\
&b_{12}c_{21} - b_{21}c_{12} = 0 \\
&b_{11}c_{12} + b_{12}c_{22} - b_{12}c_{12} - b_{22}c_{12} = 0 \\
&b_{21}c_{11} + b_{22}c_{21} - b_{11}c_{21} - b_{21}c_{21} = 0 \\
&b_{21}c_{12} + b_{12}c_{22} - b_{12}c_{21} - b_{22}c_{12} = 0.
\end{aligned} \tag{2.3}$$

There are no real solutions to these equations since a contradiction arises when all of the elements of the matrices for e_2 or e_3 are required to vanish. The three-dimensional representation of the Heisenberg group is

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.4}$$
$$e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 2.1: There exists no real two-dimensional representation of the Sol algebra.

Proof:

For the algebra with commutation relations

$$\begin{aligned}
&[e_1, e_2] = 0 \\
&[e_1, e_3] = -e_2 \\
&[e_2, e_3] = e_1
\end{aligned} \tag{2.5}$$

Let

$$e_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$
$$e_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \tag{2.6}$$
$$e_3 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

Then

$$[e_1, e_2] = \begin{bmatrix} b_1c_2 - b_2c_1 & a_1b_2 + b_1d_2 - a_2b_1 - b_2d_1 \\ c_1a_2 + d_1c_2 - c_2a_1 - d_2c_1 & c_1b_2 - c_2b_1 \end{bmatrix}$$
$$[e_2, e_3] = \begin{bmatrix} b_2c_3 - b_3c_2 & a_2b_3 + b_2d_3 - a_3b_2 - b_3d_2 \\ c_2a_3 + d_2c_3 - c_3a_2 - d_3c_2 & c_2b_3 - c_3b_2 \end{bmatrix} \tag{2.7}$$
$$[e_3, e_1] = \begin{bmatrix} b_3c_1 - b_1c_3 & a_3b_1 + b_3d_1 - a_1b_3 - b_1d_3 \\ c_3a_1 + d_3c_1 - c_1a_3 - d_1c_3 & c_3b_1 - c_1b_3 \end{bmatrix}$$

yielding the equations

$$\begin{aligned}
&b_1c_2 - b_2c_1 = 0\\
&a_1b_2 + b_1d_2 - a_2b_1 - b_2d_1 = 0\\
&c_1a_2 + d_1c_2 - c_2a_1 - d_2c_1 = 0\\
&b_2c_3 - b_3c_2 = 0\\
&a_2b_3 + b_2d_3 - a_3b_2 - b_3d_2 = b_1\\
&c_2a_3 + d_2c_3 - c_3a_2 - d_3c_2 = c_1\\
&b_2c_3 - b_3c_2 = - d_1\\
&b_3c_1 - b_1c_3 = a_2\\
&a_3b_1 + b_3d_1 - a_1b_3 - b_1d_3 = b_2\\
&c_3a_1 + d_3c_1 - c_1a_3 - d_1c_3 = c_2\\
&c_3b_1 - c_1b_3 = d_2.
\end{aligned} \tag{2.8}$$

Setting $a_1= -d_1$ and $a_2 = -d_2$, it follows that

$$\begin{aligned}
&b_1c_2 = b_2c_1\\
&a_1b_2 = a_2b_1\\
&a_1c_2 = a_2c_{1.}
\end{aligned} \tag{2.9}$$

Then the first two generators are

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} \tag{2.10}$$

$$\begin{bmatrix} \lambda a_1 & \lambda b_1 \\ \lambda c_1 & -\lambda a_1 \end{bmatrix}$$

By the commutation relations

$$\begin{aligned}
&2\lambda\, a_1\, b_3 + \lambda\, b_1d_3 - \lambda\, a_3\, b_1 = b_1\\
&- (2a_1b_3+b_1d_3-a_3b_1) = \lambda\, b_1.
\end{aligned} \tag{2.11}$$

Consequently, $\lambda^2 = - 1$ and $\lambda \pm i$. There exists no real two-dimensional generators of the generators of the Sol algebra.

Theorem 2.2: There exists a three-parameter set of real three-dimensional representations of the Sol algebra.

Proof.
Let

$$\begin{aligned}
e_1 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\\
e_2 &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\\
e_3 &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}
\end{aligned} \tag{2.12}$$

The commutation relations produce the following equations for a_{ij}, b_{ij} and c_{ij}

$$\begin{aligned}
&a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}-b_{11}a_{11}-b_{12}a_{21}-b_{13}a_{31} = 0\\
&a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}-b_{21}a_{11}-b_{22}a_{21}-b_{23}a_{31} = 0\\
&a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}-b_{31}a_{11}-b_{32}a_{21}-b_{33}a_{31} = 0\\
&a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}-b_{11}a_{12}-b_{12}a_{22}-b_{13}a_{32} = 0\\
&a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}-b_{21}a_{12}-b_{22}a_{22}-b_{23}a_{32} = 0\\
&a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}-b_{31}a_{12}-b_{32}a_{22}-b_{33}a_{32} = 0\\
&a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}-b_{11}a_{13}-b_{12}a_{23}-b_{13}a_{33} = 0\\
&a_{21}b_{13}+a_{12}b_{23}+a_{13}b_{33}-b_{11}a_{12}-b_{12}a_{23}-b_{13}a_{33} = 0\\
&a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}-b_{31}a_{13}-b_{32}a_{23}-b_{33}a_{33} = 0\\
&a_{11}c_{11}+a_{12}c_{21}+a_{13}c_{31}-c_{11}a_{11}-c_{12}a_{21}-c_{13}a_{31} = -b_{11}
\end{aligned} \tag{2.13}$$

$$\begin{array}{l}
a_{21}c_{11}+a_{22}c_{21}+a_{23}c_{31}-c_{21}a_{11}-c_{22}a_{21}-c_{23}a_{31} = -b_{21} \\
a_{31}c_{11}+a_{32}c_{21}+a_{33}c_{31}-c_{31}a_{11}-c_{32}a_{21}-c_{33}a_{31} = -b_{31} \\
a_{11}c_{12}+a_{12}c_{22}+a_{13}c_{32}-c_{11}a_{12}-c_{12}a_{22}-c_{13}a_{32} = -b_{12} \\
a_{21}c_{12}+a_{22}c_{22}+a_{23}c_{32}-c_{21}a_{12}-c_{22}a_{22}-c_{23}a_{32} = -b_{22} \\
a_{31}c_{12}+a_{32}a_{22}+a_{33}c_{32}-c_{31}a_{12}-c_{32}a_{22}-c_{33}a_{32} = -b_{32} \\
a_{11}c_{13}+a_{12}c_{23}+a_{13}c_{33}-c_{11}a_{13}-c_{12}c_{23}-c_{13}a_{33} = -b_{13} \\
a_{21}c_{13}+a_{22}c_{23}+a_{23}c_{33}-c_{21}a_{13}-c_{22}a_{23}-c_{23}a_{33} = -b_{23} \\
a_{31}c_{13}+a_{32}c_{23}+a_{33}c_{33}-c_{31}a_{13}-c_{32}a_{23}-c_{33}a_{33} = -b_{33}
\end{array}$$

The homogeneous equations can be satisfied by $b_{ij}= \lambda\, a_{ij}$ for some constant λ. If the relations had been linearly independent, giving nine equations for nine unknowns, this would be the only solution. However, it is possible to select a set of four elements from $\{a_{12}, a_{13}, a_{21}, a_{31}, b_{12}, b_{13}, b_{21}, b_{31}\}$ such that the first relation is valid trivially.

For example, let a_{12}, b_{31}, a_{21}, b_{12} equal zero. The first expression vanishes, and remaining homogeneous equations are

$$\begin{array}{l}
a_{22}b_{31} - b_{21}a_{11} = 0 \\
a_{31}b_{11} + a_{32}b_{21} - b_{32}a_{21} - b_{33}a_{31} = 0 \\
a_{11}b_{12} + a_{13}b_{32} - b_{12}a_{22} - b_{13}a_{32} = 0 \\
a_{23}b_{32} - b_{23}a_{32} = 0 \\
a_{31}b_{12} + a_{32}b_{22} - b_{32}a_{22} - b_{23}a_{32} = 0 \\
a_{13}b_{33} - b_{11}a_{13} - b_{12}a_{23} - b_{13}a_{33} = 0 \\
a_{22}b_{23} - b_{32}a_{23} = 0.
\end{array} \tag{2.14}$$

The fourth and eighth equations are identical. Setting $a_{11}=a_{22}$, $a_{12}b_{32}=0$ follows from the third equation.

If $a_{13}=0$, $b_{12}a_{23}=0$. Let $b_{12}=0$. Then

$$\begin{array}{l}
a_{32}b_{22} - b_{32}a_{22} - b_{33}a_{32} = 0 \\
a_{11}b_{33} - b_{11}a_{13} = 0 \\
b_{11} = b_{33} \\
a_{32}b_{21} = 0
\end{array} \tag{2.15}$$

Let $b_{21} = 0$ and

$$\begin{array}{l}
a_{32}b_{22} - b_{32}a_{22} - b_{33}a_{32} = 0 \\
b_{13}a_{33} = 0 \\
a_{22}b_{23} + a_{23}b_{33} - b_{22}a_{23} - b_{23}a_{33} = 0
\end{array} \tag{2.16}$$

One solution to Eq. (2.16) is

$$\begin{array}{ll}
a_{32} = 0 & b_{32}=0 \\
a_{22} = a_{33} & a_{23}=b_{13}=b_{23}=0.
\end{array} \tag{2.17}$$

The matrices representing e_1 and e_2 would have the form

$$\left(\begin{matrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{22} \end{matrix}\right)~~~~~~~~~~\left(\begin{matrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & b_{32} & b_{11} \end{matrix}\right) \tag{2.18}$$

The next set of conditions is

$$\begin{array}{l}
-c_{13}a_{31} = -b_{11} \\
(a_{22}-a_{11})c_{21}-c_{23}a_{31}-b_{21} = 0 \\
a_{31}c_{11}+a_{22}c_{31}-c_{31}a_{11}-c_{33}a_{31} =-b_{31} = 0 \\
a_{11}c_{12}-c_{12}a_{22}-c_{13}a_{32} =-b_{12} \\
a_{31}c_{12}+a_{22}c_{32}-c_{32}a_{22} = -b_{32} \\
a_{11}c_{13}-c_{13}a_{22} = -b_{13} = 0 \\
a_{22}c_{23}-c_{23}a_{33} = -b_{23} = 0 \\
a_{33}c_{13} = -b_{13} = -b_{11}
\end{array} \tag{2.19}$$

Then

$$b_{12}c_{21} + b_{13}c_{31} - c_{12}b_{21} - c_{13}b_{31} = 0 = a_{11}. \tag{2.20}$$

The above set of conditions is abbreviated to

$$
\begin{array}{lll}
c_{13}a_{31} = b_{11} & & \\
a_{22}c_{21}-c_{23}a_{31} = 0 & a_{31}c_{23}=0 & c_{23}=0 \\
a_{31}c_{11}+a_{22}c_{31}-c_{33}a_{31}= 0 & c_{11}=c_{33} & \\
c_{12}a_{22} = b_{21} = 0 & a_{22}=0 & \\
a_{31}c_{12} = -b_{32} & & \\
c_{13}a_{22} = 0 & & \\
a_{22}c_{13} = -b_{11} = 0 & &
\end{array}
\tag{2.21}
$$

The first two matrices now have the form

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32} & 0 \end{bmatrix}
$$

The subsequent relations yield

$$
\begin{array}{ll}
b_{32}c_{21} = a_{31} & \\
-c_{13}b_{32} = a_{12} = 0 & c_{13} = 0 \\
-c_{23}b_{32} = a_{22} = 0 & c_{23} = 0 \\
b_{32}(c_{22}-c_{33}) = a_{32} = 0 & c_{22}=c_{33} \\
a_{13} = a_{23} = 0 & \\
b_{32}\, c_{23} = a_{33} = 0 & c_{23} = 0
\end{array}
\tag{2.23}
$$

The elements of the matrix for e_3 are

$$
\begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{11} & 0 \\ c_{31} & c_{32} & 0 \end{bmatrix}
\tag{2.24}
$$

The commutators $[e_1, e_3] = - e_2$ and $[e_2, e_3] = e_1$ yield the equalities

$$
\begin{array}{l}
a_{31} = b_{32}c_{21} \\
a_{31}c_{12} = -b_{32}.
\end{array}
\tag{2.25}
$$

A set of matrices which satisfies the commutator relations of the Sol algebra

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} c_{11} & -1 & 0 \\ 1 & c_{11} & 0 \\ c_{31} & c_{32} & c_{11} \end{bmatrix}
\tag{2.26}
$$

provides a realization of the three-dimensional geometry. Since c_{11}, c_{31} and c_{32} have been left undetermined, there is a three-parameter set of generators.

Since more conditions are derived from the commutation relations of the three-dimensional Lie algebras of the other basic geometries, the minimum dimension for a real representation again will be three.

It follows that the Lie algebra generators can be regarded as vector fields in $\mathbf{R}^3$. Consider, for example, the Nil algebra with the generators

$$
\begin{array}{l}
e_1 = \partial/\partial y - x\, \partial/\partial z \\
e_2 = \partial/\partial x \\
e_3 = \partial/\partial z.
\end{array}
\tag{2.27}
$$

Theorem 2.3: The solutions to $e_1e_3\ \Phi = (\partial/\partial y - x\, \partial/\partial z)\, \partial/\partial z\ \Phi=0$ invariant under the isometry group in the Nil geometry form a subset of the space of functions $f(\hat{y}+\hat{z})+g(\hat{y}-\hat{z})$ where

$$\hat{y} = (2/ (x^2+ \sqrt{x^4+4})^{1/2}\ (1/N_1\, y+ 1/ N_2 z)$$
$$\hat{z} = (2/(\sqrt{x^4+4}-x^2)^{1/2}\, (1/N_1(1- (x^2+ \sqrt{x^4+4})/2)y+ 1/ N_2\, (1-(x^2- \sqrt{x^4+4})/2)z)$$
$$N_1 = [1+1/ x^2\, (1-(x^2+\sqrt{x^4+4})/2))^2]^{1/2}$$
$$N_2 = [1+ 1/x^2\, (1-(x^2+\sqrt{x^4+4})/2)^2)^2]^{1/2}.$$

Proof:
Since $[e_1, e_3]=0$, $(e_1-e_3)(e_1+e_3)=(e_1+e_3)(e_1-e_3)$ and

$$((\partial/\partial y - x\,\partial/\partial z) - \partial/\partial z))((\partial/\partial y - x\,\partial/\partial z) + \partial/\partial z) = ((\partial/\partial y - x\,\partial/\partial z) + \partial/\partial z)((\partial/\partial y - x\,\partial/\partial z) - \partial/\partial z)$$
$$= \partial^2/\partial y^2 - 2x\,\partial^2/\partial y\,\partial z + (x^2-1)\,\partial^2/\partial z^2 \qquad (2.28)$$
$$= (\partial/\partial y\ \partial/\partial z)' \begin{bmatrix} 1 & -x \\ -x & x^2-1 \end{bmatrix} \begin{bmatrix} \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

The eigenvalues of the matrix $\begin{bmatrix} 1 & -x \\ -x & x^2-1 \end{bmatrix}$ are

$\lambda_1=(x^2+\sqrt{x^4+4})/2$ and $\lambda_2 = (x^2-\sqrt{x^4+4})/2$ with eigenvector

$$v_1= 1/N_1 \begin{bmatrix} 1 \\ 1/\ x\ (1-\ \left(x^2+\ \sqrt{x^4+4}\right)/2 \end{bmatrix}$$

$$N_1= [1+1/x^2\,(1-(x^2+\sqrt{x^4+4})/2\,)^2]^{1/2} \qquad (2.29)$$

$$v_2= 1/ N_2 \begin{bmatrix} 1 \\ 1/\ x\ (1-\ \left(x^2-\ \sqrt{x^4+4}\right)/2 \end{bmatrix}$$

$N_2= [1+1/x^2\,(1-(x^2-\sqrt{x^4+4})/2\,)^2]^{1/2}$.

The diagonalized differential operator is given by

$$(x^2+\sqrt{x^4+4})/2\ \partial/\partial y'^2 + (x^2 - \sqrt{x^4+4}\,)/2\ \ \partial/\partial z'^2$$
$$\partial/\partial y' = 1/N_1\,\partial/\partial y+1/N_2\ \ \partial/\partial z \qquad (2.30)$$
$$\partial/\partial z' = 1/N_2\,1/x\,(1-(x^2+\sqrt{x^4+4})/2)\,\partial/\partial y + 1/N_2\,1/x\,(1-(x^2-\sqrt{x^4+4})/2)\,\partial/\partial z$$

Suppose that

$$\hat{y} = (2/(x^2+\sqrt{x^4+4}))^{1/2}\ y' \qquad (2.31)$$
$$\hat{z} = (2/(\sqrt{x^4+4}-x^2)^{1/2}\ \ z'$$

The solutions to

$$(\partial^2/\partial\hat{y}^2 - \partial^2/\partial\hat{z}^2)\,\Phi(\hat{y},\hat{z}) = 0 \qquad (2.32)$$

are $f(\hat{y}+\hat{z})+g(\hat{y}-\hat{z})$. Over the space of solutions, the vector field representation of $(e_1-e_3)(e_1+e_3)$ is zero.

Similarly,

$$[e_1^2, e_2] = e_1[e_1, e_2]+[e_1,e_2]e_1 = e_1e_3+e_3e_1 = 2e_1e_3 \qquad (2.33)$$

The vanishing of e_1e_3 therefore vanishes from that of e_1^2. By the commutator,

$$[[e_1^2, e_2],e_2]=2[e_1e_2,e_2]=2e_1[e_3,e_2]+2[e_1,e_2]e_3=2e_3^2 \qquad (2.34)$$

the vanishing of e_3^2 also vanishes from the vanishing of e_1^2.

Since $e_1^2 \in T_o(\mathrm{Nil})$ at the origin o,

$$e_1^2= \alpha\, e_1+\beta\, e_2+ \gamma\, e_3 \qquad (2.35)$$

Since

$$[e_1^2, e_2] = \alpha\,[e_1, e_2] = \alpha\, e_3, \qquad (2.36)$$

it would follows that $2e_1e_3= \alpha\, e_3$ and $\alpha=0$. The vanishing of $[e_1^2, e_1] = \beta[e_2, e_1] = -\beta e_3$ yields $\beta=0$. Then

$e_1^2 e_2 = \gamma\, e_3$. However, $e_1e_2= \delta e_3$. Then

$$e_1(e_2- \delta/\gamma\, e_1)=0 \qquad (2.37)$$

for $\gamma\neq 0$.

Since e_1 and e_2 are not proportional, $\gamma = 0$, and $e_1^2 = 0$.

Finally, let $e_2^2= \alpha'\, e_1+\beta'\, e_2+ \gamma' e_3 \qquad (2.38)$

The commutators

$$[e_2^2, e_1] = \beta' [e_2, e_1] = \beta' e_3 \qquad (2.39)$$
$$[e_2^2, e_1] = e_2[e_2,e_1]+[e_2,e_1]e_2 = -e_2e_3+(-e_3)e_2=-2e_2e_3$$

require $\beta' = 0$. By the vanishing of $[e_2^2, e_2] = \alpha'[e_1, e_2]= \alpha' e_3=0$ and $\alpha'=0$. Then $e_2^2= \gamma' e_3$ and

$$(e_1- \delta /\gamma' e_2) e_2=0. \qquad (2.40)$$

for $\gamma' \neq 0$. Then γ' must be set equal to zero and $e_2^2 = 0$. The generators of the Nil algebra are nilpotent of order 2. Since e_1e_3 is zero if (e_1+e_3) (e_1-e_3) vanishes by the commutation relations, the solutions to $(\partial/\partial y - x\, \partial/\partial z)\partial/\partial z\, \Phi(y, z) = 0$ invariant under the Nil group must form a subset of the solutions to $(e_1^2-e_3^2)\Phi(y, z) = 0$ given by $\{f(\hat{y}+\hat{z}) +g(\hat{y}- \hat{z})\}$.

3. THE LIE ALGEBRAS OF BASIC FOUR-GEOMETRIES

The transformation groups of the basic four-geometries may be described similarly to the Bianchi classification of homogeneous three-dimensional geometries [2]. The commutators of the vector fields spanning these spaces would follow from the relations

$$\begin{aligned}
&[e_1, e_2] = c_{123}e_3+c_{124}e_4\\
&[e_1, e_3] = c_{132}e_2+c_{134}e_4\\
&[e_2, e_3] = c_{231}e_1+c_{234}e_4\\
&[e_1, e_4] = c_{142}e_2+c_{143}e_3\\
&[e_2, e_4] = c_{241}e_1+c_{243}e_3\\
&[e_3, e_4] =c_{341}e_1+c_{342}e_2.
\end{aligned} \qquad (3.1)$$

Then a set of double commutators is

$$\begin{aligned}
&[[e_1, e_2], e_3] = c_{124} [e_4, e_3] = -c_{124}(c_{341}e_1+c_{342}e_2)\\
&[[e_2, e_3], e_1] = c_{234} [e_4, e_1] = -c_{234}(c_{142}e_2+c_{143}e_3)\\
&[[e_3, e_1], e_2] = -c_{134} [e_4,e_2] = c_{134}(c_{241}e_1+c_{243}e_3)
\end{aligned} \qquad (3.2)$$

By the Jacobi identity.

$$\begin{aligned}
&-c_{124} (c_{341}e_1+c_{342}e_2)-c_{234}(c_{142}e_2+c_{143}e_3)+c_{134}(c_{241}e_1+c_{243}e_3)\\
&\quad = - (c_{124}c_{341}-c_{134}c_{241})e_1- (c_{124}c_{342}+c_{234}c_{142})e_2 - (c_{234}c_{143}-c_{134}c_{243}) e_3\\
&\quad = 0
\end{aligned} \qquad (3.3)$$

or

$$\begin{aligned}
&c_{124}c_{341} = c_{134}c_{241}\\
&c_{124}c_{342} = -c_{234}c_{142}\\
&c_{234}c_{143} = c_{134}c_{243}.
\end{aligned} \qquad (3.4)$$

Similarly, it follows from the commutators

$$\begin{aligned}
&[[e_1, e_2], e_4] = c_{123}[e_3, e_4] = c_{123}(c_{341}e_1+c_{342}e_2)\\
&[[e_4, e_1], e_2] = - c_{143}[e_3, e_2] = c_{143}(c_{231}e_1+c_{234}e_4)\\
&[[e_2, e_4], e_1] = c_{243}[e_3, e_1] = -c_{243}(c_{132}e_2+c_{134}e_4)
\end{aligned} \qquad (3.5)$$

or

$$c_{123}(c_{341}e_1+c_{342}\, e_2)+c_{143}(c_{231}e_1+c_{234}e_4) - c_{243}(c_{132}e_2+c_{134}e_4) = 0, \qquad (3.6)$$

which requires the inequalities

$$\begin{aligned}
&c_{123}c_{341} = - c_{143}c_{231}\\
&c_{123}c_{342} = c_{243}c_{132}\\
&c_{143}c_{234} = c_{243}c_{134}.
\end{aligned} \qquad (3.7)$$

The commutation relations

$$\begin{aligned}
&[[e_1, e_3], e_4] = c_{132}[e_2, e_4] = c_{132}(c_{241}e_1+c_{243}e_3) \qquad (3.8)\\
&[[e_4, e_1], e_3] = - c_{142}[e_2, e_3] = - c_{142}(c_{231}e_1+c_{234}e_4)\\
&[[e_3, e_4], e_1] = c_{342}[e_2, e_1] = - c_{342}(c_{123}e_3+c_{124}e_4)
\end{aligned}$$

satisfy the Jacobi identity if

$$(c_{132}c_{241}-c_{142}c_{231})e_1+(c_{132}c_{243}-c_{234}\, c_{123})e_3-(c_{142}c_{234}+c_{342}c_{124})e_4 = 0 \qquad (3.9)$$

or

$$\begin{aligned}
&c_{132}c_{241} = c_{142}c_{231}\\
&c_{132}c_{243} = c_{342}c_{123}\\
&c_{142}c_{234} = - c_{342}c_{124}.
\end{aligned} \qquad (3.10)$$

The commutators

$$\begin{aligned}
&[[e_2, e_3], e_4] = c_{231}[e_1, e_4] = c_{231}(c_{342}e_2+c_{143}e_3)\\
&[[e_4, e_2], e_3] = - c_{241}[e_1,e_3] = - c_{241}(c_{132}e_2+c_{134}e_4)\\
&[[e_3, e_4], e_2] = c_{341}[e_1, e_2] = c_{341}(c_{123}e_3+c_{124}e_4)
\end{aligned} \qquad (3.11)$$

yield the equality

$$(c_{231}c_{342}-c_{241}c_{132})e_2 + (c_{231}c_{143}+c_{341}c_{123})e_3 + (c_{341}c_{124}-c_{241}c_{134})e_4 = 0 \qquad (3.12)$$

or

$$c_{231}c_{342} = c_{241}c_{132}$$
$$c_{231}c_{143} = - c_{341}c_{123} \quad (3.13)$$
$$c_{341}c_{124} = c_{241}c_{134}.$$

These conditions will be satisfied by the structure constants of the Lie algebra of the basic four-geometry.

Let $\gamma_{\alpha\beta}$ be the metric of a hypersurface in a four-dimensional Bianchi cosmology and

$$\Pi_{\alpha\beta\mu\nu}=c^{\rho}{}_{\alpha\beta}\ c^{\sigma}{}_{\mu\nu}\ \gamma_{\rho\sigma} \quad (3.14)$$

Then the quadratic forms [3][4]

$$q_1 = \Pi_{\alpha\beta\mu\nu}\ \gamma^{\alpha\mu}\ \gamma^{\beta\nu} \quad (3.15)$$
$$q^2= c^{\alpha}{}_{\beta\kappa}\ c^{\beta}{}_{\alpha\lambda}\ \gamma^{\kappa\lambda}$$

can be defined. Diagonalizing the positive-definite metric $\gamma_{\rho\sigma}$, $q^1> 0$, The structure constants determine the compactness of the three-dimensional isometry group, since $g_{\kappa\lambda}= c^{\alpha}{}_{\beta\kappa}\ c^{\beta}{}_{\alpha\lambda}$ is negative-definite for compact semisimple groups, and $q^2 = g_{\kappa\lambda}\ \gamma^{\kappa\lambda} < 0$.

The reduction to three-dimensional commutation relations is sufficient to ensure the embeddability of the basic three-geometries in the basic four-geometries. The condition of inclusion in the group SO(9) [5] will place further constraints on the Lie algebra. Nevertheless, those four-geometries that do satisfy can be projected to basic three-geometries with an isometry group which is a subgroup of G_2 [6] can be included in the path integral for quantum gravity [7].

4. RICCI FLOW ON NIL GEOMETRY

A Ricci soliton will be described on the Nil geometry.
Let

$$g(t) = A(t)(\theta^1)^2 + B(t)(\ \theta^2)^2 + C(t)(\ \theta^3)^2$$
$$\theta^1= dy \quad (4.1)$$
$$\theta^2 = dx$$
$$\theta^3 = xdy+dz$$

Suppose that

$$(\vartheta^1)^2 = A(t)(\theta^1)^2$$
$$(\vartheta^2\ )^2 = B(t)(\theta^2)^2 \quad (4.2)$$
$$(\vartheta^3)^2 = C(t)(\theta^3)^2$$

such that

$$g(t) = (\vartheta^1)^2 + (\vartheta^2\ )^2 + (\vartheta^3)^2. \quad (4.3)$$

By evaluating covariant derivatives with respect to the frame

$$F_1=1/\sqrt{A(t)}\ e_1= 1/\ \sqrt{A(t)}\ \ (\partial/\partial y\text{-}x\ \partial/\partial z)$$
$$F_2=1/\sqrt{B(t)}\ e_2= 1/\ \sqrt{B(t)}\ \ \partial/\ \partial x \quad (4.4)$$
$$F_3= 1/\ \sqrt{C(t)}\ e_3= 1/\ \sqrt{C(t)}\ \ \partial/\partial z$$

Since $[F_1,F_2]= 1/\ \sqrt{(A(t)B(t))}\ e_3= \sqrt{C(t)/(A(t)B(t))}F_3$,

$$R_{11}(g(t)) = -1/2\ C(t)/B(t)$$
$$R_{22}(g(t)) = -1/2\ C(t)/A(t) \quad (4.5)$$
$$R_{33}(g(t)) = ½\ C(t)^2/A(t)B(t).$$

The Ricci flow equations would be

$$\partial/\partial t\ \ g(t)_{11} = \ C(t)/B(t)$$
$$\partial/\partial t\ g(t)_{22} = C(t)/A(t) \quad (4.6)$$
$$\partial/\partial t\ g(t)_{33} = -\ C(t)^2/(A(t)B(t)).$$

Then

$$d/dt(A(t)B(t)C(t)) = B(t)C(t)\ dA(t)/dt +A(t)C(t)dB(t)/dt + A(t)B(t)dC(t)/dt$$
$$= C^2(t) \quad (4.7)$$

When C(t) increases in magnitude, $A(t)B(t)C(t) \rightarrow \infty$.

If $A(t) \rightarrow -A(t)$ for the Lorentzian metric $-(\vartheta^1)^2 + (\vartheta^2\ )^2 + (\vartheta^3)^2$,

$$d/dt\ (A(t)) =-C(t)/B(t)$$
$$d/dt(B(t))= -C(t)/A(t) \quad (4.8)$$
$$d/dt\ (C(t))=C2(t)/A(t)B(t)$$

and

$$\frac{d}{dt}(A(t)B(t)C(t)) = -C^2(t). \quad (4.9)$$

by combining the three equations [8]. When $C(t) \to \infty$, $A(t)B(t)C(t) \to 0$, and the Lorentzian soliton [9] decreases in volume with time.

The expansion of the Euclidean Ricci soliton prevents its embedding within a four-sphere of fixed radius. However, if the time coordinate occurs in the line element with the opposite sign, then the ambient space-time may be de Sitter space. The Euclidean Ricci soliton might be embedded in the hyperbolic space $\mathbf{H}^4$. A matching of the metrics

$$ds_{\text{Nil soliton}}^2 = A^2(t)(dy+xdz)^2+B^2(t)dx^2+C^2(t)dz^2 \quad (4.10)$$

$$ds_{\mathbf{H}4}^{\ 2}\,|_{\text{constant }\eta} = 1/\eta^2[d\eta^2+dx^2+dy^2+dz^2]\,|_{\text{constant }\eta}$$

The line element for the soliton in the Nil geometry equals

$$ds_{\text{Nil soliton}}^2 = [\ dy\ dz\ ]\begin{bmatrix} A^2(t) & xA^2(t) \\ xA^2(t) & A^2(t)x^2+C^2(t)\end{bmatrix}\begin{bmatrix} dy \\ dz\end{bmatrix} \quad (4.11)$$

The eigenvalues of the matrix are

$$\lambda_1 = (A^2(t)(1+x^2)+C^2(t))^{1/2}\,[1+[1+(4x^2\,A^4(t))/(A^2(t)(1+x^2)+C^2(t))^2]^{1/2}] \quad (4.12)$$

$$\lambda_2 = (A^2(t)(1+x^2)+C^2(t))^{1/2}\,[1-[1+(4x^2\,A^4(t))/(\,A^2(t)(1+x^2)+C^2(t))^2]^{1/2}]$$

and the eigenvectors are

$$v_1 = \begin{bmatrix} v_{21} \\ v_{12}\end{bmatrix}$$

$$= 1/N_1{}'\begin{bmatrix} 1 \\ -1/\ x+1/2x\ \left(A^2(t)\left(1+x^2\right)+C^2(t)\right)\ [1+[1+\left(4x^2A^4(t)\right)/\ \left(A^2(t)\left(1+x^2\right)+C^2(t)\right)^2]^{1/\ 2}\end{bmatrix}$$

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22}\end{bmatrix}$$

$$= 1/N_1{}'\begin{bmatrix} 1 \\ -1/\ x+1/2x\ \left(A^2(t)\left(1+x^2\right)+C^2(t)\right)\ [1-[1+\left(4x2A4(t)\right)/\ \left(A2(t)(1+x2)+C2(t)\right)2]1/\ 2\ ]\end{bmatrix}$$

$$N_1{}' = [1+[-1/x+1/2x\,(A^2(t)(1+x^2)+C^2(t))\,[1+[1+(4x^2A^4(t))/(A^2(t)(1+x^2)+C^2(t))^2]^{1/2}\,]^2] \quad (4.13)$$

$$N_2{}' = [1+[-1/x+1/2x\,(A^2(t)(1+x^2)+C^2(t))\,[1-[1+(4x^2\,A^4(t))/(A^2(t)(1+x^2)+C^2(t))^2]^{1/2}]^2$$

Since

$$\begin{bmatrix} v_1{}^T \\ v_2{}^T\end{bmatrix}\begin{bmatrix} A^2(t) & xA^2(t) \\ xA^2(t) & A^2(t)x^2+C^2(t)\end{bmatrix}\begin{bmatrix} v_1 & v_2\end{bmatrix} = \begin{bmatrix} \lambda_1 v_1{}^T v_1 & \lambda_2 v_1{}^T v_2 \\ \lambda_1 v_2{}^T v_1 & \lambda_2\, v_2{}^T\, v_2\end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \quad (4.14)$$

$$\begin{bmatrix} A^2(t) & x\,A^2(t) \\ xA^2(t) & A^2(t)x+C^2(t)\end{bmatrix} = \begin{bmatrix} v_1 & v_2\end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix} v_1{}^T \\ v_2{}^T\end{bmatrix} \quad (4.15)$$

Then

$$\begin{bmatrix} dy & dz\end{bmatrix}\begin{bmatrix} v_1 & v_2\end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix} v_1{}^T \\ v_2{}^T\end{bmatrix}\begin{bmatrix} dy \\ dz\end{bmatrix} = \begin{bmatrix} dy' & dz'\end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix} dy' \\ dz'\end{bmatrix} \quad (4.16)$$

with

$$dy' = dy\ v_{11}+dz\ v_{21} \quad (4.17)$$

$$dz' = dy\ v_{12}+dz\ v_{22}$$

Then

$$ds^2 = B^2(t)dx^2+\lambda_1(t)dy'^2+\lambda_2(t)\,dz'^2. \quad (4.18)$$

Let

$$\hat{y}(t) = \lambda_1{}^{1/2}(t)/B(t)\ y' \quad (4.19)$$

$$\hat{z}(t) = \lambda_2{}^{1/2}(t)/B(t)\ z'$$

such that

$$ds^2 = B^2(t)\,(\,dx^2+d\hat{y}^2+d\hat{z}^2\,) \quad (4.20)$$

Choosing η_0 = constant=$1/B(t_0)$, the Euclidean Ricci soliton at time t_0 may be embedded in $\mathbf{H}^4$. For a continuous range of values of η and t, with $\eta = 1/B(t)$, there is a continuous embedding of this soliton in a constant η slice of the hyperboloid.

The spatial coordinates expand, however, only at a linear rate. Given a linear increase in C(t), the equation d/dt (A(t)B(t)C(t)) = $C(t)^2$ would allow a linear increase in A(t) and B(t). By contrast, the Lorentz Ricci soliton could be embedded in a four-manifold of fixed volume if its signature is changed. It would be necessary to have the time coordinate to have the same sign as two of spatial coordinates in the signature. Then the interpretation of the remaining spatial coordinate reflects an interchange with time, except that forward time then would correspond to a decrease in this direction.

5. CONCLUSION

The requirement of three dimensions for a real representation of the Lie algebras determining the symmetry groups of the basic three-geometries indicates that a projection to two dimensions must introduce complex numbers. The transition from Poisson brackets of classical variables to quantum commutation relations requires the imaginary unit. Therefore, the compatibility of the transition from two to three dimensions with geometric group invariance may require quantum theory.

The embedding of the Ricci flow, which have determined the diffeomorphism classes of three-manifolds, in four dimensions appears to require a change in the signature if the volume of the solition exceeds certain bounds. The physical theories in three dimensions, therefore, areentirely described in four dimensions under certain conditions such as Lorentz signature of the manifolds.

REFERENCES

1. W. P. Thurston, Three-Dimensional Groups, Kleinian Geometry and Hyperbolic Geometry, Bulletin Amer. Math. Soc. 6 (1982) 357-381.
2. M. A. H. MacCallum, On the Classification of the Real Four-DimensionalLie Algebras, in *On Einstein's Path*, ed. A. Harvey, Springer, New York, 1999,pp. 299-317.
3. M. A. Akivis and V. V. Goldberg, Conformal Differential Geometry andits Generalizations, J. Wiley, New York, 1996.
4. T. Christodoulakis, E. Korfiatis and G. O. Papadopoulos, Automorphism Inducing Diffeomorphisms, Invariant Characterization of Homogeneous 3-Spaces and Hamiltonian Dynamics of Bianchi Cosmologies,Commun. Math. Phys. 226 (2002) 377-391.
5. S. Davis, Parallelizability of the Basic Four-Geometries, RFSC-15-28.
6. R. Albuquerque and I. M. C. Salavessa, The G2-Sphere over a 4-Manifold, Mon. Math. 158 (2009) 335-348.
7. S. Davis, The Path Integrals for String Theory and Quantum Gravity: Three-Geometries and Conformal Embeddings in Four-Manifolds, RFSC-09-24.
8. K. Onda, Lorentz Ricci Solitons on 3-Dimensional Lie Groups, Geometricae Dedicata 147 (2010) 313-322.
9. M. Brozos-V{\'a}zquez, G. Calvaruso, E. Garc{\'i}a-R{\'i}o and S. Gavino-Fernandez, Three-Dimensional Lorentzian Homogeneous Ricci Solitons, Isr. J. Math. 188 (2012) 385-403.

Source of Support: Nil, Conflict of interest: None Declared