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A FIELD ALGEBRA STRUCTURE ON THE REPRESENTATION OF THE AFFINE LIE ALGEBRA $D_n^{(1)}$

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ABSTRACT

In this paper, the vertex operator structure on the representation of the affine algebra associated with $D_n^{(1)}$ is studied by the representation theory of Lie algebra. Then, it is proved that the representation V_Q can be constructed a field algebra according to calculus methods of formal distributions, and the conformal vector on the field algebra is given.

Key words: *vertex operator; field algebra; n-th product.*

1. INTRODUCTION

The physicists brought forward the concept of vertex operation algebra in studying the theory of field and string. It is important in studying representation theory and finite group. Meurmen and Lepowsky sloved the Guss of Mckay-Thompson with this theory. And Borcherds used the vertex operation algebra and Kac-Moody Lie algebra to slove the famous problem of the Monstrous Moonishine Conjecture and won fields award in 1998. Frenkel and Kac[1,2] had constructed the level-one representations of affine Kac-Moody algebras $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8$ by means of vertex operators in 1981.In addition, Xu and Jiang [3] have introduced another set of vertex operators in1990 which are constructed for the level-one representations of the cases $B_n^{(1)}$ and $G_2^{(1)}$. Xu[4] gave the level-one representations of the affine Lie algebras with first kind in 1991.

It is an important subject of using these vertex operators to construct a field algebra. The representations V_Q of Kac-Moody Lie algebra associated to $D_n^{(1)}$ are constructed, which are based on a certain untwisted or twisted vertex operators. In this paper, we use the vertex operators of affine Lie algebra $D_n^{(1)}$ with first kind to construct a field algebra. Since the Jacobi Identity for the definition of vertex operator algebra is very complicated, we instead it with the axiom of weak locality to construct field algebra.

2. THE VERTEX REPRESENTATION V_Q OF $D_n^{(1)}$

In this section, we briefly introduce the structure of V_Q and vertex operators $Y(v,z)$ on V_Q.

Let $B(x,y)$ be the Killing form of a finite dimensional complex simple Lie algebra $SO(2n,\mathbb{C})$. Let $\mathfrak{H}$ ($\dim(\mathfrak{H})=n$) be a Cartan subalgebra of $SO(2n,\mathbb{C})$ and Δ be the root system. Then $SO(2n,\mathbb{C})=\mathfrak{H}+\sum_{\alpha\in\Delta} g_\alpha$, g_α is the root subspace decomposition of by the Cartan subalgebra $\mathfrak{H}$. Let $\Pi=\{\alpha_1,\cdots,\alpha_n\}\subset\mathfrak{H}^*$ be a simple root system, where $\mathfrak{H}^*$ is the dual space of $\mathfrak{H}$.

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The root lattice

$$L = \{\ \alpha = \sum_{i=1}^{n} m_i \alpha_i \mid m_1, \cdots, m_n \in \mathbb{Z}\ \} \subset \mathfrak{H}_R^* \tag{1}$$

is an abelian addition group in the real linear space $\mathfrak{H}_R^*$. Then $\mathfrak{H}_R^*$ has an inner product $(x, y) = c_0 B(x, y)$, $\forall\ x, y \in \mathfrak{H}_R^*$, where c_0 is a positive constant. The group algebra $\mathbb{C}(e^L)$ of L is an abelian associative algebra with the basis $\{e^{\alpha} \mid \alpha \in L\}$, where $e^0 = 1$ and $e^{\alpha} e^{\beta} = e^{\alpha+\beta}$.

Supposed $h_i(m) = t^m \otimes \alpha_i,\ m \in \mathbb{Z}, 1 \le i \le n,\ X_m(\alpha) = t^m \otimes e_{\alpha},\ \forall\, \alpha \in \Delta,$ where t is a complex parameter. Let S^- be the complex linear space spanned by the basis $1, h_i(-m), m \in \mathbb{Z}^+, 1 \le i \le n$. Denote $S(S^-)$ be the symmetric tensor algebra over $\mathbb{C}$ generated by S^- with the product $\vee$. Then $S(S^-)$ is a commutative associated algebra with the unit element 1 and has a basis

$$1, h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s),\ 1 \le i_1 \le \cdots \le i_s \le n, m_1, \cdots, m_s \in \mathbb{Z}^+, s \in \mathbb{Z}^+$$

Denote $V = S(S^-) \otimes C(e^L)$. The formal linear combination of finite or infinite elements of the basis forms a complete space V_Q of V. It is well known that V_Q is an associative algebra with

$$(h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s)) \otimes e^{\alpha} = (1 \otimes e^{\alpha}) \prod_{k=1}^{s} (h_{i_k}(-m_k \otimes e^0). \tag{2}$$

Hence the representation space V has a basis

$$1 \otimes e^{\beta}, (h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s)) \otimes e^{\beta} = (1 \otimes e^{\beta}) \prod_{k=1}^{s} ((h_{i_k}(-m_k) \otimes e^0,$$

where $\beta \in L, 1 \le i_1 \le \cdots \le i_s \le n, m_1, \cdots, m_s \in \mathbb{Z}^+, s = 1, 2, \cdots.$

For any $u \otimes e^{\beta} \in V_Q$, the degree of $u \otimes e^{\beta}$ is defined by

$$\deg(u \otimes e^{\beta}) = \deg(u) + \frac{1}{2}(\beta, \beta),$$

where $\deg(u)$ is defined by

$$\deg(1) = 0,\ \deg[h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s)] = \sum_{k=1}^{s} m_k.$$

Then we introduce some linear operators acting on V for the definition of the vertex operator representation of the affine Lie algebra, vertex operator algebra and vertex algebra.

(I) Let $D : V_Q \to V_Q$ be a linear operator, which is defined by

$$D(v \otimes e^{\beta}) = \deg(v \otimes e^{\beta})(v \otimes e^{\beta}). \tag{2}$$

(II) Let $\partial_{h_i(m_i)}$, $1 \le i \le n$, $m_i \in \mathbb{Z}$ be the linear differential operators acting on the linear space $S(S^-)$, which are defined by

$$\partial_{h_i(m_i)}(h_j(-m_j)) = m_i \delta_{m_i, -m_j}(\alpha_i, \alpha_j), m_i, m_j \in \mathbb{Z}^+. \tag{3}$$

Let $\alpha_i(m_i)$ be a linear operator acting on V_Q, which is defined by

$$\begin{cases} \alpha_i(-m_i)(v \otimes e^{\beta}) = (h_i(-m_i) \vee v) \otimes e^{\beta}, & \text{when } m_i \in \mathrm{Z}^+, \\ \alpha_i(0)(v \otimes e^{\beta}) = (\alpha_i, \beta)(v \otimes e^{\beta}), & \\ \alpha_i(m_i)(v \otimes e^{\beta}) = \partial_{h_i(m_i)}(v) \otimes e^{\beta}, & \text{when } m_i \in \mathrm{Z}^+. \end{cases}$$

where $\mathrm{v} \in S(S^-), e^{\beta} \in C(e^L)$.

Lemma 1: $[\alpha_i(m_i), \alpha_j(q_j)] = m_i \delta_{m_i, -q_j}(\alpha_i, \alpha_j)\mathrm{id}, m_i, q_j \in \mathbb{Z}.$

Proof: This formula has been easily proved in Refs.[6,7].

Let $\alpha = \sum_{i=1}^{n} a_i \alpha_i \in L$. The linear operator $\alpha(m)$ acting on V is defined by $\alpha(m) = \sum_{i=1}^{n} a_i \alpha_i(m), \forall\, m \in \mathbb{Z}$. By the induction, we have

Lemma 2: Let $\alpha(m)=\sum_{i=1}^{n} a_i\alpha_i(m), \beta(q)=\sum_{j=1}^{n} b_j\alpha_j(q)$, then

$$[\alpha(m),\beta(q)]=m\delta_{m+q,0}(\alpha,\beta)\mathrm{id} \tag{4}$$

Particularly,

$$\exp(\alpha(m))\exp(\beta(-m)=\exp(m(\alpha,\beta))\exp(\beta(-m))\exp(\alpha(m))$$

(III) The mapping $\varepsilon:\ L\times L\to\{e^{i\theta}\,|\,0\le\theta<2\pi\}$ is called the ε - mapping, if ε satisfies the following conditions:

(i) $\varepsilon(0,\beta)=\varepsilon(\beta,0)=1,\forall\beta\in L;$

(ii) $\varepsilon(\alpha,\beta)=(-1)^{(\alpha,\beta)}\varepsilon(\beta,\alpha),\quad \forall\alpha,\beta\in L;$

(iii) $\varepsilon(\beta+\gamma,\alpha)\varepsilon(\beta,\gamma)=\varepsilon(\beta,\alpha+\gamma)\varepsilon(\gamma,\alpha),\forall\alpha,\beta,\gamma\in L.$

(IV) Let $x(\alpha,z)$: $\mathbb{C}(e^L)\to\mathbb{C}(e^L)$ be a linear operator, which is defined by

$$x(\alpha,z)(v\otimes e^{\gamma})=\varepsilon(\alpha,\gamma)z^{(\alpha,\gamma)}(v\otimes e^{\alpha+\gamma}) \tag{5}$$

Let $X(\alpha,z)=\sum_{m=-\infty}^{\infty}X_m(\alpha)z^{-m}$ be the Laurent series of z. Then

$$X(\alpha,z)(v\otimes e^{\gamma})=E^{+}(\alpha,z)E^{-}(\alpha,z)x(\alpha,z)(v\otimes e^{\gamma})=\mathrm{ex}\sum_{m=1}^{\infty}\mathrm{p}\frac{z^{m}}{m}\alpha(-m)\mathrm{ex}\sum_{m=1}^{\infty}\mathrm{p}\frac{-z^{-m}}{m}\alpha(m)\varepsilon(\alpha,\gamma)z^{(\alpha,\gamma)}(v\otimes e^{\alpha+\gamma}).$$

Theorem 1: The vertex operator representation (ρ,V) of affine Lie algebras with the first kind can be defined on the generators by

$$\begin{cases}\rho(c)=\mathrm{id},\\ \rho(d)=D,\\ \rho(t^m\otimes\alpha_i)=\alpha_i(m),\quad 1\le i\le n,\quad m\in Z,\\ \rho(t^m\otimes e_\alpha)=X_m(\alpha),\quad \alpha\in\Delta,\quad m\in Z.\end{cases}$$

This theorem is proofed in Pre[8]. In this case, the multiplication table of the affine Lie algebra is

(1) $[\mathrm{id},D]=[\mathrm{id},\alpha_i(m)]=[\mathrm{id},X_m(\alpha)]=0,[D,\alpha_i(m)]=m\alpha_i(m),$

(2) $[D,X_m(\alpha)]=mX_m(\alpha),[\alpha_i(m),\alpha_j(k)]=m\delta_{m,-k}(\alpha_i,\alpha_j)\mathrm{id},$

(3) $[X_m(\alpha),X_k(-\alpha)]=\varepsilon(\alpha,-\alpha)(\alpha(m+k)+m\delta_{m,-k}\mathrm{id}),$

(4) $[X_m(\alpha),X_k(\beta)]=0,\alpha,\beta\in\Delta,\alpha+\beta\notin\Delta\cup\{0\},\ [X_m(\alpha),X_k(\beta)]=\varepsilon(\alpha,\beta)X_{m+k}(\alpha+\beta),\alpha,\beta,\alpha+\beta\in\Delta$.

3. THE STRUCTURE OF VERTEX ALGEBRA ON V_Q

Definition 1: A complex linear space V is called a field algebra, if there exist a set of linear operators (every linear operator is called a field) for v:

$$Y(v,z)=\sum_{m\in\mathbb{Z}}v_{(m)}z^{-m-1}\in\mathrm{End}V[[z,z^{-1}]], \tag{6}$$

such that given any $v,w\in V$, there is a positive integer $m_0=m_0(v,w)$, such that $v_{(m)}(w)=0,\forall m>m_0$. And there is a fixed vector $|0>\in V$, which is called by the vacuum vector, such that

(i) (vacuum)

$$Y(|0>,z)=\mathrm{id}_V,\quad Y(v,z)|0>|_{z=0}=v.$$

(ii) (translation convariance)

$T\in\mathrm{End}(V)$ is defined by $T(v)=v_{(-2)}|0>,\forall v\in V$. T is called the infinitesimal translation operator, if T is a derivation on V and satisfies the condition: $\mathrm{ad}(T)=\partial_z$ acting on any linear operator $Y(u\otimes e^{\gamma},z)$.

(iii) (weak locality) There exists a positive integer N, such that

$$\mathrm{Res}_z(z-w)^N[Y(u,z),Y(v,w)]=0,\ \forall u,v\in V.$$

This definitions is same in Pre. [9, 10]

Define the map

$$Y(\cdot,z):V_Q\longrightarrow(EndV_Q)[[z,z^{-1}]]$$

$$v\longrightarrow\sum_{n\in\mathbb{Z}}v_nz^{-n-1}$$

by the following way:

(1) $Y(1,z)=Id_{V_Q}$; $Y(h_i(-1)\otimes 1,z)=H_i(z)=\sum_{m\in\mathbb{Z}}H_i(m)Z^{-m-1}$;

(2) $Y(h_i(-m)\otimes 1,z)=\partial^{(m-1)}H_i(z)$; $Y(1\otimes e^{\alpha})=X(\alpha,z)=E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z)$;

(3) $Y(h_{i_1}(-m_1)\vee\cdots\vee h_{i_s}(-m_s)\otimes 1,z)=:\partial^{(m_1-1)}H_{i_1}(z)\cdots\partial^{(m_s-1)}H_{i_s}(z):$

(4) $Y(h_{i_1}(-m_1)\vee\cdots\vee h_{i_s}(-m_s)\otimes e^{\beta},z)=:\partial^{(m_1-1)}H_{i_1}(z)\cdots\partial^{(m_s-1)}H_{i_s}(z)Y(1\otimes e^{\beta},z):$

where $:\cdot:$ is the normal order of fields or operators, $\partial^{(m)}=\frac{\partial^m}{m!}$ is a differential operator, and the vacuum vector $|0>=1\otimes e^0$.

In the following, we will check that $(V,Y(v\otimes e^{\alpha},z))$ is a field algebra, i.e., it satisfies the three axioms of the vertex algebra.

3.1. The vacuum axiom

(1) $Y(1\otimes 1,z)=Id_v$;

(2) $Y(h_i(-m)\otimes 1,z)1\otimes 1|_{z=0}=h_i(-m)\otimes 1$;

(3) $Y(v\otimes e^{\alpha},z)1\otimes 1|_{z=0}=v\otimes e^{\alpha}$.

This formula can be easily proved by the Theorem 1.

3.2. The translation covariance axiom

Let $v=\prod_{k=1}^{s}h_{i_k}(-m_k)$, $v_k=\prod_{1\le j\le s,j\ne k}h_{i_j}(-m_j)$, $m_i\in\mathbb{Z}^+$.Then

$$T(v\otimes e^{\gamma})=\gamma(-1)(v\otimes e^{\gamma})+\sum_{k=1}^{s}m_k h_{i_k}(-(1+m_k))(v_k\otimes e^{\gamma}).$$

T is a derivation which acts on V_Q. Particularly,

$$T(1\otimes e^{\gamma})=\gamma(-1)(1\otimes e^{\gamma}),\ T(h_i(-m)\otimes 1)=mh_i(-m-1)\otimes 1.$$

Lemma 3: Given $i=1,2,\cdots,n$, $m\in\mathbb{Z}_+, m>0,\ \alpha\in L$, then

(i) $\operatorname{ad}(T)\alpha(-m)=m\alpha(-(1+m))$;

(ii) $\operatorname{ad}(T)\alpha(1)(v\otimes e^{\alpha+\gamma})=-(\alpha,\alpha+\gamma)(v\otimes e^{\alpha+\gamma})$;

(iii) $\operatorname{ad}(T)\alpha(m)=-m\alpha(m-1)$;

(iv) $\operatorname{ad}(T)x(\alpha,z)=\alpha(-1)x(\alpha,z)$;

(v) $\operatorname{ad}(T)E^+(\alpha,z)=E^+(\alpha,z)\sum_{m=2}^{\infty}z^{m-1}\alpha(-m)$;

(vi) $\operatorname{ad}(T)E^-(\alpha,z)=[\sum_{m=1}^{\infty}z^{-m-1}\alpha(m)+z^{-1}(\alpha,\alpha+\gamma)]E^-(\alpha,z);$

$$E^-(\alpha,z)[T,x(\alpha,z)](v\otimes e^{\gamma})=[\alpha(-1)-z^{-1}(\alpha,\alpha)]E^-(\alpha,z)x(\alpha,z)(v\otimes e^{\gamma}).$$

Proof: The formulas (i-iii,v) is easily proved[11]. Now, we only check the formulas (iv) and (v).

$$\begin{aligned}\operatorname{ad}(T)E^+(\alpha,z)&=\operatorname{ad}(T)\prod_{m=1}^{\infty}\exp\frac{1}{m}z^m\alpha(-m)\\&=\sum_{j=1}^{\infty}\prod_{m=1}^{j-1}\exp\frac{1}{m}z^m\alpha(-m)\operatorname{ad}(T)\bullet\exp\frac{1}{j}z^j\alpha(-j)\prod_{m=j+1}^{\infty}\exp\frac{1}{m}z^m\alpha(-m)\\&=E^+(\alpha,z)\sum_{j=1}^{\infty}z^j\alpha(-(1+j)).\end{aligned}$$

Hence (iv)holds.

Lemma 4: The infinitesimal translation operator T satisfies the translation covariance axioms

$$[T,Y(v\otimes e^{\gamma})]=\partial_z Y(v\otimes e^{\gamma}).\quad (7)$$

Proof: Since ∂_z is a differential operator acting on $Y(u\otimes e^{\alpha})$ about z,then

$$\partial_z(Y(1\otimes e^{\alpha},z))(v\otimes e^{\gamma})$$
$$=\partial_z(E^{+}(\alpha,z)E^{-}(\alpha,z)x(\alpha,z))(v\otimes e^{\gamma})$$
$$=\partial_z(E^{+}(\alpha,z))E^{-}(\alpha,z)x(\alpha,z)+E^{+}(\alpha,z)\partial_z(E^{-}(\alpha,z))x(\alpha,z)+E^{+}(\alpha,z)E^{-}(\alpha,z)\partial_z(x(\alpha,z))(v\otimes e^{\gamma})$$
$$=E^{+}(\alpha,z)[\sum_{m=1}^{\infty}z^{m-1}\alpha(-m)+\sum_{m=1}^{\infty}z^{-m-1}\alpha(m)+z^{-1}(\alpha,\gamma)]E^{-}(\alpha,\sqrt{z})(\partial_z\tilde{x}(\alpha,\sqrt{z}))(v\otimes e^{\gamma}).$$

By the formulas (iv), (v) and (vi) of Lemma 3, we have

$$[T\ Y(1\otimes e^{\alpha},z)](v\otimes e^{\gamma})$$
$$=[T\ E^{+}(\alpha,z)E^{-}(\alpha,z)x(\alpha,z)](v\otimes e^{\gamma})$$
$$=[[T\ E^{+}(\alpha,z)]E^{-}(\alpha,z)x(\alpha,z)+E^{+}(\alpha,z)[T\ E^{-}(\alpha,z)]x(\alpha,z)+E^{+}(\alpha,z)E^{-}(\alpha,z)[T\ x(\alpha,z)]](v\otimes e^{\gamma})$$
$$=E^{+}(\alpha,z)[\sum_{m=1}^{\infty}z^{m-1}\alpha(-m)+\sum_{m=1}^{\infty}z^{-m-1}\alpha(m)+z^{-1}(\alpha,\gamma)]E^{-}(\alpha,z)x(\alpha,z)(v\otimes e^{\gamma}),$$

Therefore

$$[T\ Y(1\otimes e^{\alpha},z)]=\partial_z(Y(1\otimes e^{\alpha},z)).$$

It is easy to obtain that

$$[T\ Y(h_i(-m)\otimes e^{0},z)]=\partial_z(Y(h_i(-m)\otimes e^{0},z))$$

by the formulas (ii), (iv)and (vi) of Lemma 3. By the induction, we have

$$[T\ Y(v\otimes e^{0},z)]=\partial_z(Y(v\otimes e^{0},z)),$$

and

$$[T\ Y(u\otimes e^{\alpha},z)]=\partial_z(Y(u\otimes e^{\alpha},z)),$$

where $u=\prod_{k=1}^{s}h_{i_k}(-m_k)$. Then

$$\mathrm{ad}\,(T)Y(h_{i_0}(-m_0\vee u\otimes e^{\alpha},z)=\mathrm{ad}\,(T):\partial^{(m_0-1)}H_{i_0}(z)Y(u\otimes e^{\alpha}):$$
$$=\mathrm{ad}\,(T)[\partial^{(m_0-1)}H_{i_0}(z)_{+}Y(u\otimes e^{\alpha})]+\mathrm{ad}\,(T)[Y(u\otimes e^{\alpha})\partial^{(m_0-1)}H_{i_0}(z)_{-})]$$
$$=[\mathrm{ad}\,(T)\partial^{(m_0-1)}H_{i_0}(z)_{+}]Y(u\otimes e^{\alpha})+\partial^{(m_0-1)}H_{i_0}(z)_{+}[\mathrm{ad}\,(T)Y(u\otimes e^{\alpha})]$$
$$+[\mathrm{ad}\,(T)Y(u\otimes e^{\alpha})]\partial^{(m_0-1)}H_{i_0}(z)_{-}+Y(u\otimes e^{\alpha})[\mathrm{ad}\,(T)\partial^{(m_0-1)}H_{i_0}(z)_{-}]$$
$$=\partial_z Y(h_{i_0}(-m_0)\vee u\otimes e^{\alpha})]+[(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{+}]Y(u\otimes e^{\alpha})+Y(u\otimes e^{\alpha})[(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{-}].$$

Hence we need to prove that

$$[(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{+}]Y(u\otimes e^{\alpha})+Y(u\otimes e^{\alpha})[(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{-}]=0.$$

From the formulas (i) and (iii), it is easy to prove

$$[(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{+}]=0,\ [(\mathrm{ad}\,(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_{-}]=0.$$

Hence (10) holds.

3.3. The weak locality axiom

In this subsection, we check the weak locality axiom. Namely, we will prove the following formula holds.

$$\mathrm{Res}_z(z-w)^N[Y(u\otimes e^{\alpha},z),Y(v\otimes e^{\beta})]=0,\quad \forall\ u\otimes e^{\alpha},v\otimes e^{\beta}\in V_Q,z,w\in\mathbb{C}.$$

Lemma 5: $Y(h_i(-1)\otimes 1,z)$ and $Y(h_j(-1)\otimes 1,w)$ are weak local, i.e.,

$$\mathrm{Res}_z(z-w)^N[Y(h_i(-1)\otimes 1,z),Y(h_j(-1)\otimes 1,w)]=0.$$

Proof: Notice that

$$[Y(h_i(-1)\otimes 1,\ Y(h_j(-1)\otimes 1,w)]=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}[H_i(m)\ H_j(n)]z^{-m-1}w^{-n-1}=(\alpha_i,\beta_j)\sum_{m\in\mathbb{Z}}mz^{-m-1}w^{-m-1}=(\alpha_i,\beta_j)\partial_w\delta(z-w).$$

By the formula (iv) of Lemma 3, it is easy to prove

$$(z-w)^2[Y(h_i(-1)\otimes 1\ Y(h_j(-1)\otimes 1,w)]=(\alpha_i,\beta_j)(z-w)^2\partial_w\delta(z-w)=0.$$

It is followed that

$$\mathrm{Res}_z(z-w)^2[Y(h_i(-1)\otimes 1\ Y(h_j(-1)\otimes 1,w)]=0.$$

Lemma 6[12]: $Y(h_i(-m_i)\otimes 1,z)$ and $Y(h_j(-n_j)\otimes 1,z)$ are weak local, i.e.,

$$\mathrm{Res}_z (z-w)^{m_i+n_j}[Y(h_i(-m_i)\otimes 1,z),Y(h_j(-n_j)\otimes 1,w)]=0 .$$

Proof: Since

$$Y(h_i(-m_i)\otimes 1,z)\cdot Y(h_j(-n_j)\otimes 1,w)=Y(h_j(-n_j)\otimes 1,w)\cdot Y(h_i(-m_i)\otimes 1,z)$$
$$+(\alpha_i,\alpha_j)\sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j}\cdot\frac{(m+m_i-1)!}{m!(m_i-1)!}\frac{(m+m_i+n_j-1)!}{(m-m_i)!(n_j-1)!}(-m-m_i)a_{n_j}(n_j)$$
$$+(\alpha_i,\alpha_j)\sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n\cdot\frac{(n+m_i+n_j-1)!}{(n+n_j)!(m_i-1)!}\frac{(n+n_j-1)!}{n!(n_j-1)!}(n+n_j)a_{m_i}(m_i),$$

then

$$Y(h_i(-m_i)\otimes 1,z)\cdot Y(h_j(-n_j)\otimes 1,w)-Y(h_j(-n_j)\otimes 1,w)\cdot Y(h_i(-m_i)\otimes 1,z)$$
$$=(\alpha_i,\alpha_j)(\sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n\frac{(n+m_i+n_j-1)!}{(m_i-1)!n!(n_j-1)!}a_{m_i}(m_i)-\sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j}\frac{(m+m_i+n_j-1)!}{(n_j-1)!m!(m_i-1)!}a_{n_j}(n_j))$$
$$=(\alpha_i,\alpha_j)\frac{(m_i+n_j-1)!}{(m_i-1)!(n_j-1)!}[\sum_{n=0}^{\infty} z^{-n-m_i-n_j} w^n\cdot\frac{(n+m_i+n_j-1)!}{n!(m_i+n_j-1)!}a_{m_i}(m_i)-\sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j}\frac{(m+m_i+n_j-1)!}{m!(m_i+n_j-1)!}a_{n_j}(n_j)]$$

By the formula (iv) of Lemma 3, we have

$$(z-w)^{m_i+n_j}[Y(h_i(-m_i)\otimes 1,z),Y(h_j(-n_j)\otimes 1,w)]=0 .$$

It is followed that

$$\mathrm{Res}_z (z-w)^{m_i+n_j}[Y(h_i(-m_i)\otimes 1,z),Y(h_j(-n_j)\otimes 1,w)]=0 .$$

Lemma 7: $Y(h_i\otimes 1,z)$ and $Y(1\otimes e^{\alpha},w)$ are weak local, i.e.,

$$\mathrm{Res}_z (z-w)^N[Y(h_i\otimes 1,z),\, Y(1\otimes e^{\alpha},w)]=0 .$$

Proof: Since

$$Y(h_i\otimes 1,z)=H_i(z)=\sum\nolimits_{m\in\mathbb{Z}} H_i(m)z^{-m-1},$$

then

$$[H_i(m),\, Y(1\otimes e^{\alpha},w)]=w^m(\alpha_i,\,\alpha)Y(1\otimes e^{\alpha},w),$$

and

$$[Y(h_i\otimes 1,z)\; Y(1\otimes e^{\alpha},w)]=\delta(z-w)Y(1\otimes e^{\alpha},w).$$

Therefore,

$$(z-w)[Y(h_i\otimes 1,z)\; Y(1\otimes e^{\alpha},w)]=0 .$$

It is followed that

$$\mathrm{Res}_z (z-w)[Y(h_i\otimes 1,z)\; Y(1\otimes e^{\alpha},w)]=0 .$$

Lemma 8: $Y(1\otimes e^{\alpha},z)$ and $Y(1\otimes e^{\beta},w)$ are weak local, i.e.

$$\mathrm{Res}_z (z-w)^N[Y(1\otimes e^{\alpha},z),\, Y(1\otimes e^{\beta},w)]=0 .$$

Proof: Since

$$Y(1\otimes e^{\alpha},z)=E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z),$$

we have

$$Y(1\otimes e^{\alpha},z)Y(1\otimes e^{\beta},w)$$
$$=x(\alpha,z)x(\beta,w)E^+(\alpha,z)E^-(\alpha,z)E^+(\beta,z)E^-(\beta,z)$$
$$=z^{-(\alpha,\beta)}(z-w)^{(\alpha,\beta)}x(\alpha,z)x(\beta,w)E^+(\alpha,z)E^+(\beta,z)E^-(\alpha,z)E^-(\beta,z).$$

By the same way, we get

$$Y(1\otimes e^{\beta},z)Y(1\otimes e^{\alpha},w)$$
$$=x(\beta,w)x(\alpha,z)E^+(\beta,w)E^-(\beta,w)E^+(\alpha,z)E^-(\alpha,z)$$
$$=w^{-(\alpha,\beta)}(w-z)^{(\alpha,\beta)}x(\beta,w)x(\alpha,z)E^+(\alpha,z)E^+(\beta,z)E^-(\alpha,z)E^-(\beta,z).$$

From the definition of mapping ε [13], we have

$$z^{-(\alpha,\beta)}(z-w)^{(\alpha,\beta)}x(\alpha,z)x(\beta,w)=w^{-(\alpha,\beta)}(w-z)^{(\alpha,\beta)}x(\beta,w)x(\alpha,z),$$

i.e.

$$(z-w)[Y(1\otimes e^{\alpha},z),Y(1\otimes e^{\beta},w)]=0.$$

then

$$\mathrm{Res}_z(z-w)[Y(1\otimes e^{\alpha},z),Y(1\otimes e^{\beta},w)]=0.$$

Lemma 9[13]: If $a(z)$, $b(z)$ and are pairwise mutually local fields, then $a(z)_{(n)}b(z)$ and $c(z)$ are mutually local fields for all $n\in Z$ (resp. $n\in Z_+$).In particular $:a(z)b(z):$ and $:c(z):$ are mutually local fields provided that $a(z)$, $b(z)$ and $c(z)$ are.

Lemma 10: For any $u\otimes e^{\alpha}$, $v\otimes e^{\beta}\in V_Q$,there exists a non-negative integer N satisfies

$$\mathrm{Res}_z(z-w)^N[Y(u\otimes e^{\alpha},z),Y(v\otimes e^{\beta},w)]=0.$$

Proof: From the Lemma 5-8,we know that $Y(h_i(-1)\otimes 1,z)$, $Y(h_j(-m)\otimes 1)$ and $Y(1\otimes e^{\alpha})$ are pairwise mutually local fields. Then by the Lemma 9 ,we can easily prove Lemma 10.
From the fact that we have checked that $(V,Y(v\otimes e^{\alpha},z))$ satisfies the three axioms. It follows

Theorem 2: $(V,Y(v\otimes e^{\alpha},z))$ is a field algebra.

4. THE CONFORMAL VECTOR OF THE FIELD ALGEBRA

Definition 3: A conformal vector of a vertex algebra V is an even vector v such that the corresponding fie $Y(v,z)=\sum_{n\in Z}L_n z^{(-n-2)}$ is a Virasoro field with cetral charge c which has the following properties:
(a) $L_{-1}=T$,
(b) L_0 is diagonalizable on V .
The number c is called the central charge of v .
Let $A=((\alpha_i,\alpha_j))^{-1}=(a_{ij})$, then A is a symmetric matrix. Let

$$v=\frac{1}{2}\sum_{k=1}^{n}a_{jk}H_j(-1)H_k(-1)|0>,$$

then

$$Y(v,z)=\frac{1}{2}\sum_{j,k=1}^{n}a_{jk}:H_j(z)H_k(z):=\sum_{n\in Z}L_n z^{(-n-2)}.$$

We have

$$L_0=\frac{1}{2}\sum_{j,k}^{n}a_{jk}H_j(0)H_k(0)+\frac{1}{2}\sum_{j,k}^{n}\sum_{n>0}a_{jk}(H_j(-n)H_k(n)+H_j(n)H_k(-n));$$

$$L_{-1}=\frac{1}{2}\sum_{j,k}^{n}\sum_{n\geq 0}a_{jk}(H_j(-n-1)H_k(n)+H_k(-n-1)H_j(n));$$

$$L_m=\frac{1}{2}\sum_{j,k}^{n}\sum_{n\in\mathbb{Z}}a_{jk}H_j(n)H_k(m-n),\ m\neq 0.$$

In the following, we will proof that the L_m satisfies the above properties (a) and (b).

By the define of L_0, we have

$$L_0(1\otimes e^{\beta})=\frac{1}{2}\sum_{j,k}^{n}a_{jk}H_j(0)H_k(0)(1\otimes e^{\beta})=\frac{1}{2}(\beta,\beta)(1\otimes e^{\beta})\tag{8}$$

$$\begin{aligned}L_0(h_i(-m)\otimes 1)&=\frac{1}{2}\sum_{j,k}^{n}\sum_{n>0}a_{jk}(H_j(-n)H_k(n)+H_j(n)H_k(-n))h_i(-m)\otimes 1\\&=\frac{1}{2}\sum_{j,k}^{n}a_{jk}(m(\alpha_k,\alpha_i)h_j(-m)\otimes 1+m(\alpha_j,\alpha_i)h_k(-m)\otimes 1)\qquad(9)\\&=mh_i(-m)\otimes 1,\end{aligned}$$

Then, by the Inductive method, so

$$L_0(h_1(-m_1)\vee\cdots\vee h_s(-m_s)\otimes e^{\beta})=(m_1+\cdots+m_s+\frac{1}{2}(\beta,\beta))(h_1(-m_1)\vee\cdots\vee h_s(-m_s)\otimes e^{\beta}).$$

Thus L_0 satisfies the above properties (a) i.e. L_0 is diagonalizable on V.

By the define of L_{-1}, we have

$$L_{-1}(1\otimes e^{\alpha})=\alpha(-1)(1\otimes e^{\alpha}),$$

and

$$L_{-1}(h(-m)\otimes 1)=m(h(-m)\otimes 1).$$

By the define of T, we proof that L_{-1} satisfies the above properties (b) i.e. $L_{-1}=T$.

The field $Y(v,z)=\sum_{n\in Z}L_n z^{(-n-2)}$ is a Virasoro field and the $(V,Y(v\otimes e^{\alpha},z))$ is a conformal vertex algebra.

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