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GERAGHTY CONTRACTION PRINCIPLE FOR DIGITAL IMAGES

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ABSTRACT

In this paper, we prove Geraghty contraction principle for digital images. Our results extend the Banach fixed point theorem for digital images. Which is an extension of the Banach contraction principle. Example is provided to illustrate our results.

Keywords: *Fixed points; Banach Contraction Principle; Geraghty contraction; Digital Image; Digital Contraction.*

AMS (2010) Mathematics Subject Classification: *47H10, 54H25, 54E35, 68U10.*

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory deals many fields of mathematics such as Functional Analysis and general topology. Digital topology is area related to features of 2D and 3D digital images using objects with topological properties.

The Banach[1] contraction mapping theorem is well known in fixed point theory. In 1973, [14] Geraghty introduced the extension to the Banach contraction principle.

$$S = \{\beta : [0, \infty) \to [0, 1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}.$$

Definition [14] Let (X, d) be a metric space. A self map $f : X \to X$ is said to be a *Geraghty contraction* if there exists $\beta \in S$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Theorem [14] Let (X, d) be a complete metric space. Let $f : X \to X$ be a *Geraghty contraction*. Then for any choice of initial point $x_0 \epsilon X$, the iteration $\{x_n\}$defined by $x_n = f(x_{n-1})$ for $n = 1, 2, 3, \ldots$ converges to the unique fixed point z of f in X

Let X be a subset of Z^n for a positive integer n where Z^n is the set of lattice points in the n - Dimensional Euclidean Space and ƙ be represent an adjacency relation for the members of X. A digital image consists (X, ƙ).

Definition 1.1: [5] Let k, m be positive integers, $1 \leq l \leq m$ and two distinct points $(p = p_1, p_2, p_3, \ldots p_m)$, $(q = q_1, q_2, q, \ldots q_m) \in Z^n$ $p$and$q$are k_l- adjacent if there are at most l indices $i$such that $|p_i - q_i| = 1$, and for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The following are the consequences above definition.

Two points p and q in Z are 2- adjacent if $|p - q| = 1$.

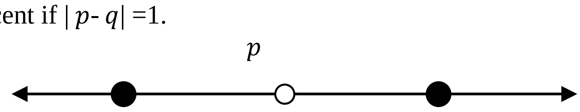

Figure-1.2: adjacent

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Two points $p$and q in Z^2 are 8- adjacent if they are distinct and differ by at most 1 in each coordinate.

Two points p and q in Z^2 are 4- adjacent if they are 8 - adjacent and differ in exactly one coordinate.

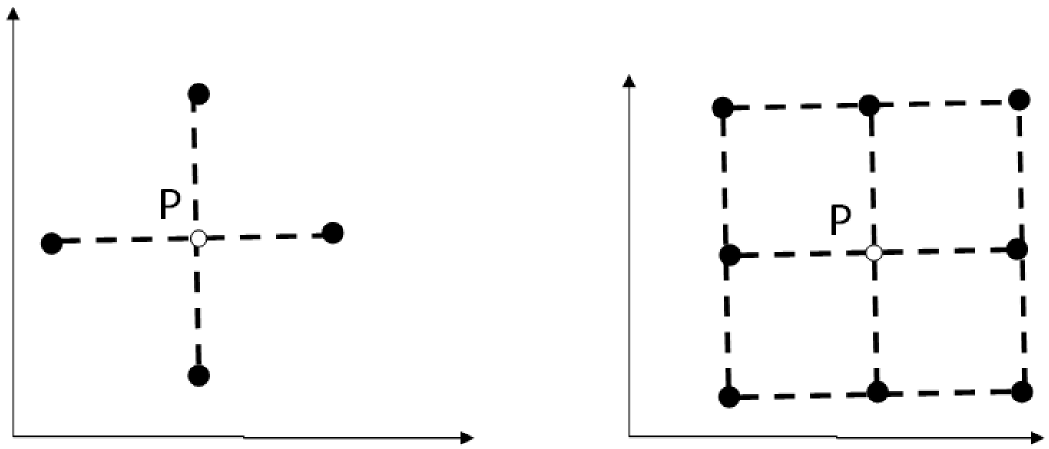

Figure- 1.3:

Two points p and q in Z^3 are 26- adjacent if they are distinct and differ by at most 1 in each coordinate.
Two points p and $q$in Z^2 are 18- adjacent if they are 26- adjacent and differ at most two coordinates.
Two points p and $q$in Z^2 are 6- adjacent if they are 18- adjacent and differ at most two coordinates. [5] A k – *neighbor* of p ∈ Z^n is a point of Z^n that is k – adjacent top where k ∈ {2,4,6,8,18,26}and n ∈ 1,2,3.

The set $N_k(p) = \{q|q\ is\ k - adjecent\ to\ p\}$ is called the k- neighborhood of p.

A digital interval is defined by $[a, b]_Z = \{ z \in Z|\ a \leq z \leq b\}$, where a, b ∈ Z $and\ \ a < b$.

A digital image X $\subset Z^n$is *k- connected* [7] if and only if for every pair of different points $x,\ y \in X,$ there is a set $\{x_0, x_1, x_2, \dots x_r\}$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are k- neighbors where i= 0,1,2,… r-1.

Definition [7]: Let $(X, k_0) \subset Z^{n0}$, $(Y, k_1) \subset Z^{n1}$ be digital images and $f: X \to Y$ be a function.
If for every $k_0 -$ connected subset U of X, $f(U)$ is a $k_1 - connected$ subset of Y, then f is said to be $(k_0, k_1) - continuous$.
f is$(k_0, k_1) - continuous$ if and only if for every $k_0 - adjacent$ points $\{x_0, x_1\}\ of\ X,$ either $f(x_0) = f(x_1)$ or $f(x_0)\ and\ \ f(x_1)$ are a $k_1 -$ adjacent in Y.

f is$(k_0, k_1) - continuous$, bijective and f^{-1} is (k_1, k_0)-continuous, then $f$is called$(k_0, k_1) - isomorphism$ and denoted by $X \cong (k_0, k_1)^Y$.

A (2- k) – continuous function $f: [0, m]_Z \to X$ such that $f(0) = x,$ and $f(m) = y$ is called a digital k- path from x to y in a digital image X. In a digital image (X, k), for every two points, if there is a k-path, then X is called k-path connected.

A simple closed k-curve of m≥ 4 points in a digital image X is a sequence $\{f(0), (1)\ , f(2),,, f(m-1)\}$of images of the k-path $f: [0, m-1]_Z \to X$ such that $f(i)$ and $f(j)$ are k-adjacent if and only if $j = i \pm mod\ m.$

A point $x \in X$ is called k-corner if x is k- adjacent to two and only two points y, z ∈ X such that y and z are k-adjacent to each other.

If y, z are not k-corners and if x is the only point k-adjacent to both y, z then we say that the k- corner is simple. X is called a generalized simple closed k-curve if what is obtained by removing all simple k-corners of X is a simple closed k-curve.

For a k-connected digital image (X, k) in Z^n, there is a following statement

$$|X|^x = N_{3^n - 1}(x) \cap X$$

$$k \in \{2n(n \geq 1), 3^n - 1(n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1(2 \leq r \leq n-1,\ \ n \geq 3)\},$$

$where\ \ C_t^n = \frac{n!}{(n-t)!t!}.$

Definition: [18] Let (X, k) be a digital image in Z^n, n≥ 3 and $\bar{X}= Z^n$- X. Then X is called a closed k- surface if it satisfies the following.

If (k, $\bar{k}$) $\in \{(k, 2n), (2n, 3^n - 1)\}$, where the k̂- adjacency is taken from (2.1) with $k \neq 3^n - 2^n - 1$ and $\bar{k}$ is the adjacency on X, then

a) For each point $x \in X$, $|X|^x$ has exactly one k-component k-adjacent to x;
b) $|\bar{X}|^x$ has exactly two $\bar{k}$- adjacent to xx; we denote by C^{xx} and D^{xx} these two components; and
c) For any point $y \in N_k(x) \cap X$, $N_{\bar{k}}(y) \cap C^{xx} \neq 0$ and $N_{\bar{k}}(y) \cap D^{xx} \neq 0$, where $N_k(x)$ means the k- neighbors of x.

Further, if a closed k –surface X does not have a simple k-point, then X is called simple.

If (k, $\bar{k}$) $= (3^n - 2^n - 1, 2n)$ then X is connected, for each point $x \in X$, $|X|^x$ is a generalized simple closed k- curve. If the image $|X|^x$ is a simple closed k-curve and the closed k-surface X is called simple.

Let (X, k) bea digital image and its subset be (A,k). (X, A) is called a digital image pair with k-adjacency and when A is a singleton set $\{x_0\}$, then (X,x_0) *is called a point digital image* .

Banach Contraction Principle for Digital Images.

Let (X, k) be a digital image and $f: (X, k) \to (X, k)$ be any (k, k) – continuous function. We say the digital image (X, k) has the fixed point property. If for every (k̂, k̂) – continuous map: $f\ X \to X$ there exists $x \in X$ such that $f(x) = x$.

The fixed point property is preserved by any digital isomorphism. It is a topological invariant.

Let (X, d, k) be denote the digital metric space with k-adjacency where d is usual Euclidean metric for Z^n.

Definition: A sequence $\{x_n\}$ of points of digital metric space (X, d, k) is said to be a Cauchy sequence if for all $\epsilon > 0$, there exists $\alpha \in N$ such that for all $n, m > 0$ then $d(x_n, x_m) < \epsilon$.

Definition: A sequence $\{x_n\}$ of points of digital metric space (X, d, k)converges to a limit $a \in X$ if for all$\epsilon > 0$, there exists $\alpha \in N$ such that for all $n > \alpha$ then $d(x_n,\ a)<\epsilon$.

Definition: A digital metric space (X, d, k) is a said be a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points (X, d, k) converges to a point a of(X, d, k).

Definition: Let $(X,\ k)$ be any digital image. A function $f: (X,\ k) \to (X,\ k)$ is called right- continuous if $f(a) = \lim_{x \to a^+} f(x)$ where $a \in X$.

Definition: [12] Let (X, d, k) be any digital metric space and $f: (X, d, k) \to (X, d, k)$ be a self digital map. If there exists $\lambda \in (0,1)$ such that for all $x, y \in X$, $d\big(f(x), f(y)\big) \leq \lambda\, d(x, y)$, then$f$ is called a digital contraction map.

Proposition: Every digital contraction map is digitally continuous.

Theorem [12]: Let(X, d, k) be a complete digital metric space which has a usual Euclidean metric in Z^n. Let $f: X \to X$be a digital contraction map. Then f has a unique fixed point, i.e., there exists a unique $u \in X$ such that $f(u) = u$.

In 2015 Ozgur Ege, Ismet Karaca generalized Banach contraction Principle as follows.

Theorem: [12] Let (X, d, k) be a complete digital metric space which has a usual Euclidean metric d in Z^n and let $f: X \to X$ be a digital self map. Assume that there exists a right- continuous real function $\gamma: [0, u] \to [0, u]$ where u is sufficiently large real number such that $\gamma(a) < a$ if a > 0, and let f satisfies

$$d\big(f(x_1), f(x_2)\big) \leq \gamma\left(d\big(x_{1,}, x_2\ \big)\right) \text{ for all} x_{1,}, x_2\ \in (X, d, k).$$

Then f has a unique fixed point u $\in (X, d, k)$ *a*nd the sequence $f^n(x)$converges to u for every $x \in X$.

Now we define digital Geraghty contraction map.

Definition: Let (X, d, k) be any digital metric space and $f: (X, d, k) \to (X, d, k)$ be a self digital map is said to be a digital Geraghty contraction map if there exists $\beta \in S$ such that $d\big(f(x), f(y)\big) \leq \beta\big(\ d(x, y)\big) d(x, y)$for all $x, y \in X$.

Here we observe that every digital contraction map is a digital Geraghty contraction map but its converse need not be true.

Now we prove Geraghty contraction theorem in digital metric spaces.

Theorem: Let (X, d, k)be a complete digital metric space with Euclidean metric d in Z^n and let $f: X \to X$ be a digital Geraghty contraction map. Then f has a unique fixed point u$\in$ (X, d, k).

Proof: Let $x_0 \in (X, d, k)$, we define the sequence $x_n = f(x_{n-1})$ for each $n \geq 1$.

If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f.

Without loss of generality, we assume that if $x_n \neq x_{n+1}$ for each n.

We have $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \beta\big(d(x_n, x_{n-1})\big)d(x_n, x_{n-1})$ (3.1)

Since $\beta \in S$,
$d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, which follows that $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of non-negative reals and so $\lim_{n\to\infty}(d(x_{n+1}, x_n))$ exists and it is r (say).

Now we show that r = 0.

If r>0 then from (3.1) we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \beta\big(d(x_n, x_{n-1})\big)d(x_n, x_{n-1})$$
$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leq \beta\big(d(x_n, x_{n-1})\big) < 1 \text{ for each } n \geq 1.$$

On letting $n \to \infty$, we get

$$1 = \lim_{n\to\infty}\left(\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})}\right) \leq \lim_{n\to\infty}\beta\big(d(x_n, x_{n-1})\big) \leq 1.$$

So that $\beta\big(d(x_n, x_{n-1})\big) \to 1$ as $n \to \infty$, that implies $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
Hence r = 0.

Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Suppose that $\{x_n\}$ not a Cauchy sequence. Then there exists $\epsilon > 0$integers

$$m(k) \text{and} n(k) \quad \text{with} \quad m(k) > n(k) > k \text{ such that } d\big(x_{m(k)}, x_{n(k)}\big) \geq \epsilon \tag{3.2}$$

we choose $m(k)$, the least positive integer satisfying$d\big(x_{m(k)}, x_{n(k)}\big) \geq \epsilon$, then we have $m(k) > n(k) > k$ with

$$d\big(x_{m(k)}, x_{n(k)}\big) \geq \epsilon, d\big(x_{m(k)-1}, x_{n(k)}\big) < \epsilon$$
$$\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)}\big)$$
$$< d\big(x_{m(k)}, x_{m(k)-1}\big) + \epsilon.$$

Since $d\big(x_{n(k)}, x_{n(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) < \epsilon$.

This implies $d\big(x_{m(k)}, x_{n(k)}\big) \to \epsilon$ as k $\to \infty$.

$$\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{n(k)+1}\big) + d\big(x_{n(k)+1}, x_{n(k)}\big)$$
$$< d\big(x_{m(k)}, x_{m(k)-1}\big) + \epsilon + d\big(x_{n(k)+1}, x_{n(k)}\big).$$

Since $d\big(x_{n(k)}, x_{n(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) < \epsilon$

This implies $d\big(x_{m(k)-1}, x_{n(k)+1}\big) \to \epsilon$ as k $\to \infty$.

$$\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) \leq d\big(x_{m(k)}, x_{m(k)+1}\big) + d\big(x_{m(k)+1}, x_{n(k)-1}\big) + d\big(x_{n(k)-1}, x_{n(k)}\big)$$
$$< d\big(x_{m(k)}, x_{m(k)+1}\big) + \epsilon + d\big(x_{n(k)+1}, x_{n(k)}\big).$$

Since $d\big(x_{m(k)}, x_{m(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \leq d\big(x_{m(k)}, x_{n(k)}\big) < \epsilon$.

This implies $d\big(x_{m(k)+1}, x_{n(k)-1}\big) \to \epsilon$ as k $\to \infty$.

$$d(x_{m(k)+1}, x_{n(k)}) = d(f(x_{m(k)}), f(x_{n(k)-1})) \leq \beta\left(d\big(x_{m(k)}, x_{n(k)-1}\big)\right) d(x_{m(k)}, x_{n(k)-1})$$
$$\frac{d(x_{m(k)+1}, x_{n(k)})}{d(x_{m(k)}, x_{n(k)-1})} \leq \beta\left(d\big(x_{m(k)}, x_{n(k)-1}\big)\right) < 1$$

On letting$k \to \infty$, we get

$$1 = \frac{\epsilon}{\epsilon} \leq \lim_{k\to\infty}\beta\left(d\big(x_{m(k)}, x_{n(k)-1}\big)\right) \leq 1.$$

So that $\beta\left(d(x_{m(k)}, x_{n(k)-1})\right) \to 1$ as $k \to \infty$,

Since $\beta \in S, d(x_{m(k)}, x_{n(k)-1}) \to 0$ as $k \to \infty$.

Hence it follows that $\epsilon = 0$, a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence in X, and since (X, d, k) is a complete digital metric space, $f^n(x)$ converges in (X, d, k).

Geraghty contraction principle can be used to image compression of a digital image.

Example: Let F_0 be a digital image, we make a copy of F_0 and glue it on the lower left vertex.

We create a copy of F_0 and glue it on lower right vertex, so that we have a new digital image of F_1 as follows.

As in the above same procedure of $F_{1,}$ we have a new digital image F_2 which is identical to $F_{1.}$

As a result, F_2 is the fixed point in this process.

Let T be a function which have F_i to $T(F_i)$.

Here we have F_2 isa fixed point of this function, by continuing this process, we get an infinite sequence of sets $\{F_n\}$, then the sequence $\{F_n\}$ converges to $F_{2.}$ It cannot be distinguished F_6 from F_2. As a result, the computer programme use F_6 instead of F_2 to be better resolution. At the same time, the programme could use F_2 in place of F_6 to determine easily some properties of digital image.

CONCLUSION

In this paper we give the digital version of Geraghty contract principle. This will be useful for digital topology and fixed point theory and to understand the better structure of digital images.

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