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EXTENSION OF GENERALIZED $\alpha - \varphi$ CONTRACTIVE MAPPING THEOREMS WITH RATIONAL EXPRESSIONS IN CONE METRIC SPACES AND APPLICATION

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ABSTRACT

The aim of this paper, we introduce and extend the notion of generalized $\alpha - \varphi$-contractive mappings involving rational expressions in cone metric spaces and state the existence and uniqueness of a fixed point for this mapping. Our results are generalizations and extend certain main result of [30]. Some examples are given to illustrate the obtained results and to show that these results are proper extensions of the existing ones. Then we apply the obtained theorem to study the existence of solutions to the nonlinear integral equation.

Keywords: *Fixed point, Cone metric space, $\alpha - \varphi$-contraction mapping, α-admissible, normal cone.*

1. INTRODUCTION

The first fundamental result on fixed point for contractive mapping was published and introduced by S. Banach [1] in 1922, which is known as Banach contraction mapping principle. It is widely recognized as influential sources in pure and applied Mathematics. A mapping$T: X \to X$, where (X, d) is a metric space, is said to be a contraction mapping, such that

$$d(Tx, Ty) \leq kd(x, y) \text{ , for all } x, y \in X \text{ with contractive constant } k \in [0,1).$$

This principle provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplinary. It has become a very popular source of existence and uniqueness theorems in different branches of Mathematical analysis.

These sources have been associated with new and generalized classes of contractive mappings. In this direction, Samet et al. [2] introduced the concept of α −admissible, α −contractive, $\alpha - \varphi -$ contractive mapping and proved fixed point results, further he also extend to the(α, β) −contractive mappings in metric spaces. Such that, several authors obtained further results (see for instance [3-17]).

In 2007, Huang and Zhang [18] introduced cone metric space , which is generalization of metric space by replacing the real numbers with ordered Banach space and obtained fixed point theorems of contractive mappings in these spaces. Subsequently, several authors have been generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal (see for instance [19-28]).

Quite recently, Kang, S. M. *et al.* [29] introduced $\alpha - \varphi$-contractive mappings in cone metric spaces and proved fixed point theorem, which is generalization of the result [2]. Further, Verma, M. *et al*. [30] generalized and extend the results [29] and proved fixed point theorem.

Thus, the purpose of this paper is to give the generalized version of $\alpha - \varphi$-Ccontractive type contraction mappings in cone metric spaces and get the fixed point theorem. Our results generalize and extend previously obtained result of [30].

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2. PRELIMINARIES

First, we introduce some standard notations and definitions which we needed them in the sequel see **([18]).**

Definition 2.1: Let E be a real Banach space and P be a subset of $E$and 0denote to the zero element in E, then P is called a cone if and only if:
(i) P is a non-empty set closed and $P \neq \{0\}$,
(ii) If a, b are non-negative real numbers and $x, y \in P$, then $ax + by \in P$,
(iii) $x \in P\ and - x \in P \Longrightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.
Given a cone P ⊂ E, we define a partial ordering$\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$(where int P denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \ => \| x \| \leq M \| y \|.$$

where M is least positive number satisfying the above is called the normal constant of P.

Definition 2.2: Let $P \subset E$ is called regular if every increasing sequence which is bounded above is convergent .That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq x_3 \ldots \ldots \ldots \leq y$ for some $y \in E$, then there exist $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if decreasing sequence which is bounded below is convergent.
Now for the following discussion assume that E is Banach space, P is a cone in E with $intP \neq \emptyset$ and$\leq$ is a partial ordering with respect to P.

Definition 2.3: Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.If the mapping $d: X \times X \longrightarrow E$ satisfies
(i) $0 < d(x, y) for\ all\ x, y \in X\ and\ (x, y) = 0$if and only if $x = y$,
(ii) $d(x, y) = d(y, x) for\ all\ x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y) for\ all\ x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space .The concept of cone metric space is more general than that of a metric space.

Example 2.4: Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$,where α ≥ 0 is a constant. Then (X, d) is a cone metric space.

Definition: 2.5: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq1}$ be a sequence in$X$. Then,
(1) $\{x_n\}_{n\geq1}$ Converges to x whenever for every$c \in E$ with $\theta \ll c$, if there is a natural Number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty}x_n = x\ or\ x_n \to x, (n \to \infty)$
(2) $\{x_n\}_{n\geq1}$is said to be a Cauchy sequence if for every $c \in E$ with θ<< c,if there is Natural number N such that $d(x_n, x_m) \ll c$ for all $n.m \geq N$.
(3) (X, d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

Lemma 2.6: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n\geq1}$ be a sequence in$X$. Then,
(a) $\{x_n\}_{n\geq1}$ Converges to x if and only if $d(x_n, x) \to 0 (n \to \infty)$.
(b) $\{x_n\}_{n\geq1}$is a Cauchy sequence if and only if $d(x_n, x_m) \to \infty (n, m \to \infty)$.

Lemma 2.7: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n\geq1}$ be a sequence in$X$. Then,
(i) The limit $\{x_n\}_{n\geq1}$ is unique. That is if $\{x_n\}_{n\geq1}$ Converges to x and$\{x_n\}_{n\geq1}$ Converges t y.
(ii) if $\{x_n\}_{n\geq1}$ Converges to x then $\{x_n\}_{n\geq1}$is a Cauchy sequence.

Lemma 2.8: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $\{x_n\}_{n\geq1}$ and$\{y_n\}_{n\geq1}$ be any two sequences in$X$with $\{x_n\}_{n\geq1} \to x$ and $\{y_n\}_{n\geq1} \to y$ as $n \to \infty$, then $d(x_n, y_n) = d(x, y)$.

Definition: 2.9[2]: Let $T: X \to X$ and $\alpha: X \times X \to [0, \infty)$, we say that T is α-admissible if

$$\alpha d(x, y) \geq 1 => \alpha d(Tx, Ty) \geq 1 \text{ for all } x, y \in X.$$

We denote withΨ the family of non -decreasing function $\varphi: [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \varphi^n < +\infty$ for each $t > 0$, where φ^n is n^{th} iterationφ.

Lemma 2.10[2]: For every $\varphi: [0, +\infty) \to [0, +\infty)$ the following holds:

If φ is non decreasing, then for each $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$ implies $\varphi(t) < t$ and $\varphi(0) = 0$.

Definition: 2.11[2]: Let $T: X \to X$ be a mapping in Metric space(X, d) is said to be an $\alpha-\varphi$ −contractive mapping, if there exists two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \Psi$ such that

$$\alpha d(x, y) d(Tx, Ty) \leq \varphi d(x, y), \text{ for all } x, y \in X.$$

Further **Kang *et al*. [29**] introduce the notion of this mapping in cone metric spaces as follows:

Definition 2.12: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be a mapping. T is called $\alpha-\varphi$ −contractive mapping, if there exists two functions $\alpha: X \times X \to [0, \infty)$and $\varphi \in \Psi$ suchthat

$$\alpha d(x, y) d(Tx, Ty) \leq \varphi d(x, y), \text{ for all } x, y \in X.$$

After that, **Verma, M. *et al.* [30]** present the notion generalized $\alpha-\varphi-contractive$ mapping in cone metric spaces and derived fixed point results for these mapping.

Definition 2.13: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be a mapping. T is said to be generalized $\alpha-\varphi$ −contractive mapping, if
There exist two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \Psi$ such that

$$\alpha d(x, y) d(Tx, Ty) \leq \varphi[M(x, y)]$$

where $(x, y) = \max\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$, for all $x, y \in X$.

Now we present and extend new notion of generalized $\alpha-\varphi$ −contractive mappings in cone metric spaces and derive fixed point results for these mappings in cone metric space.

Definition 2.13: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be a mapping. T is said to be generalized $\alpha-\varphi$ −contractive mapping, if There exist two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \Psi$ such that

$$\alpha d(x, y) d(Tx, Ty) \leq \varphi[M(x, y)]$$

where $M(x, y) = \left\{\begin{matrix} d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}, \\ \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)} \\ , \end{matrix}\right\}$ for all $x, y \in X$. (2.13)

3. MAIN RESULT

In this section we will extend and generalize of the given result of verma *et al*. [30].
Verma *et al.* [30] proved the following theorem in cone metric spaces:

Theorem 3.1: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be an $\alpha-\varphi$ −contractive mapping satisfying the following conditions:
(i) T is α-adimissible ;
(ii) There exists$x_0 \in X$such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous or
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in N$, then T has a fixed point.

Now we prove theorem 3.1 in the setting of cone metric spaces for generalized $\alpha-\varphi$contractive mapping as follows:

Theorem 3.2: Let (X, d) be a cone metric space and P be a normal cone with normal constant M. Let $T: X \to X$ be generalized $\alpha-\varphi$ −contraction mapping by definition [2.13] satisfying the following conditions:
(i) T is α-adimissible ;
(ii) There exists$x_0 \in X$such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then there exists $u \in X$ such that $T(u) = u$.

Proof: From **c**ondition (ii) there exists $x_0 \in X$ such that that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n = T^n x_0, \text{ for some } n \in N \tag{3.1}$$

If $x_n = x_{n+1}$ for some $n \in N$, then $u = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since T is α-adimissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 => \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in N \tag{3.2}$$

Now applying inequality (2.13) and (3.2), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \\ &\le \varphi[((x_{n-1}, x_n)] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= max \left\{ \begin{matrix} d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}, \\ \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}, \frac{d(x_{n-1},Tx_{n-1})\, d(x_n,Tx_n)}{1+d(x_{n-1},x_n)} \\ \frac{d(x_n,Tx_n)[1+(x_{n-1},Tx_{n-1})]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)}{d(x_{n-1},x_n)} \end{matrix} \right\} \\ &= max \left\{ \begin{matrix} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}, \\ \frac{d(x_{n-1},x_{n+1})+d(x_n,\ x_n)}{2}, \frac{d(x_{n-1},x_n)d(x_n, x_{n+1})}{1+d(x_{n-1},x_n)} \\ \frac{d(x_n,x_{n+1})[1+(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1}, x_n),d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \end{matrix} \right\} \\ &= max \left\{ \begin{matrix} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}, \\ \frac{d(x_{n-1},x_{n+1})}{2}, \frac{d(x_{n-1},x_n)d(x_n, x_{n+1})}{1+d(x_{n-1},x_n)} \\ \frac{d(x_n, x_{n+1})[1+(x_{n-1},x_n)]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1}, x_n),d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \end{matrix} \right\} \\ &\le [\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}] \end{aligned} \tag{3.4}$$

Owing to monotonicity of the functionφ, and using the inequalities (3.1) and (3.3), we get

$$d(x_n, x_{n+1}) \le \varphi max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \tag{3.5}$$

If for some $n \ge 1$, we have $d(x_{n-1}, x_n) \le d(x_n, x_{n+1})$ from (3.5) becomes

$$d(x_n, x_{n+1}) \le \varphi d(x_n, x_{n+1})$$

Which implies

$$\begin{aligned} \|d(x_n, x_{n+1})\| &\le \|\varphi d(x_n, x_{n+1})\| \\ &< \|d(x_n, x_{n+1}\|. \end{aligned}$$

Which is contradiction. Thus, for all $n \ge 1$, we have

$$\text{Max } \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) = d(x_{n-1}, x_n) \tag{3.6}$$

Using (3.3) and (3.4) we get that

$$d(x_n, x_{n+1}) \le \varphi\, d(x_{n-1}, x_n) \tag{3.7}$$

Continuing this process inductively, we obtain

$$d(x_n, x_{n+1}) \le \varphi^n (x_0, x_1) \quad \text{for all } n \in N \tag{3.8}$$

So, for $m, n \in N$ with $n > m$ we have

$$\begin{aligned} d(x_{n,}\ x_m) &\le d(x_{n,}x_{n-1}) + d(x_{n-1,}x_{n-2}) + \ldots\ldots + d(x_{m+1,}x_m) \\ &\le (\varphi^{n-1} + \varphi^{n-2} + \cdots \ldots \ldots \ldots \ldots \ldots + \varphi^m) d(x_{0,}x_1), \\ &\le \frac{\varphi^n}{1-\varphi}\, d(x_{0,}x_1) \end{aligned} \tag{3.9}$$

Since P is normal cone with normal constant, so by (3.9) we get

$$\|d(x_{n,}\ x_m)\| \le M[\frac{\varphi^n}{1-\varphi} \|d(x_{0,}x_1)\|\ ] \to 0,$$

which implies that $d(x_{0,}x_1) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in the cone metric space(X, d). Since (X, d) is complete. So there exist $u \in X$ such that

$$x_n \to u \text{ as } n \to \infty.$$

$$\text{i.e. } \lim_{n\to\infty} d(x_n, u) = 0 \tag{3.10}$$

On the other hand, since T is continuous, then we have $x_{n+1} = Tx_n \to Tu$ as $n \to \infty$. i.e.

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0 \tag{3.11}$$

From (3.10) and (3.11) and the uniqueness of the limit, we deduce that u is a fixed point of T, that is, $T(u) = u$. This completes the proof.

Example 3.3: Let us consider $X = [0,\infty)$ and $E = R^2, P = \{(x,y) \in E | x,y > 0\} \subset R^2$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, a|x-y|)$ where $a \geq 0$ is a constant. Then (X,d) be a cone metric space. We consider the continuous mapping $T: X \to X$ defined by

$$Tx = \begin{cases} 2x - \frac{3}{2}, & x > 1, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{if } x < 0. \end{cases}$$

We observe that here T is continuous and Banach contraction principle in the setting of cone metric space cannot apply.

Now we a mapping $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [0,1] \\ 0, & \textit{Otherwise} \end{cases}$$

Clearly, T is generalized α-φ contraction mapping with $\varphi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. Infect for all $x,y \in X$, we have

$$\alpha(x,y)d(Tx,Ty) \leq \frac{1}{2}d(x,y) \leq \varphi[M(x,y)], x,y \in X.$$

More over there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. For $x_0 = 1$, we have

$$\alpha((1, T(1)) = \alpha\left(1, \frac{1}{2}\right) = 1.$$

Let $x,y \in X$such that $\alpha(x,y) \geq 1$. therefore, we have$x,y \in [0,\frac{1}{4}]$. By definition T and α we have

$$Tx = \frac{x}{2} \in \left[0,\frac{1}{2}\right], Ty = \frac{y}{2} \in [0,\frac{1}{2}] \text{ and } \alpha(Tx,Ty) = 1.$$

So, T is α-admissible. Thus all hypothesis of theorem 3.2 are satisfied. Consequently, T has a fixed point.

Recall that, theorem 3.2 guarantees only the existence of fixed point but not uniqueness. In this example 0 and $\frac{3}{2}$ are two fixed points of T.

To assure that the uniqueness of fixed point, we will consider the following hypothesis. (*) for all $x,y \in X$ there exists $z \in X$such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Theorem 3.4: Theorem 3.2 yields a unique fixed point after adding hypothesis (*) to it.

Proof: Let u and v are two fixed point of T. From (*), there exist $z \in X$ such that

$$\alpha(u,z) \geq 1 \text{ and } \alpha(v,z) \geq 1 \tag{3.12}$$

Define a sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all$n \geq 0$ and $z_0 = z$. since T is α-admissible, from the above inequalities, for all $n \in N$

$$\alpha(u,z_n) \geq 1 \text{ and } \alpha(v,z_n) \geq 1 \tag{3.13}$$

Using inequalities (2.1) and (3.11), we get

$$\begin{aligned} d(u,z_{n+1}) &\leq d(Tu,Tz_n) \\ &\leq \alpha(u,z_n)d(Tu,Tz_n) \\ &\leq \varphi M(u,z_n) \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} M(u,z_n) &= \varphi\left[max\left\{\begin{matrix} d(u,z_n), d(u,Tu), d(z_n,Tz_n), \frac{d(u,Tu)+d(z_n,Tz_n)}{2}, \\ \frac{d(u,Tz_n)+d(z_n,Tu),}{2}, \frac{d(u,Tu),d(z_n,Tz_n)}{1+d(u,z)}, \frac{d(z_n,Tz_n)[1+d(u,Tu)]}{1+d(u,z)} \\ \frac{d(u,Tu),d(z_n,Tz_n)}{d(u,z)} \end{matrix}\right\}\right] \\ &\leq \max\{\mathrm{d}(u,z_n), d(u,z_{n+1}) \end{aligned} \tag{3.15}$$

Now owing to the monotonicity of φ and using inequality (3.14), we have

$$d(u,z_{n+1}) \leq \varphi\max\{d(u,z_n), d(u,z_{n+1}) \text{ for all } n \in N, \tag{3.16}$$

Without loss of generality, let$d(u,z_n) \geq 0$ for all $n \in N$. If

$$\max\{\mathrm{d}(u,z_n), d(u,z_{n+1}) = d(u,z_{n+1}). \text{ Then}$$
$$d(u,z_{n+1}) \leq \varphi\, d(u,z_{n+1}).$$
$$\|d(u,z_{n+1})\| \leq \|\varphi\, d(u,z_{n+1})\| < \|d(u,z_{n+1})\|.$$

Which is contradiction. thus we have

$$\max\{\mathrm{d}(u,z_n), d(u,z_{n+1}) = d(u,z_n) \tag{3.17}$$

In view of (3.16) and (3.17), we get

$$d(u, z_{n+1}) \leq d(u, z_n), \text{ for all } n \geq 1.$$

Continuing this process inductively, we get

$$d(u, z_n) = \varphi^n d(u, z_0), \text{ for all } n \geq 1.$$

Since P be a normal cone with normal constant$M$, we have

$$\| d(u, z_n)\| \leq M\|\varphi^n d(u, z_0\| \to 0 \text{ as } n \to \infty, \text{ which implies that } z_n \to u \text{ as } n \to \infty, \tag{3.18}$$

Similarly, we can get $z_n \to v$ as $n \to \infty$, (3.19)

Using triangular inequality, we have
$\|d(u, v)\| \to 0$ as $n \to \infty$. Thus $d(u, v) = 0$, which implies that $u = v$. Hence T has the unique fixed point .This completes the proof.

4. APPLICATION

In this section we shall apply theorem 3.2 to the two boundary value problem of second order differential equations:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t < u(t), t \in [0,1] \\ \qquad u(0) = u(1) = 0, \end{cases} \tag{3.20}$$

Where $f: [0,1] \times R \to [0. \infty)$ is a continuous decreasing function with respect to the second variable and satisfying

(i) For all $t \in [0,1)$ and $u_1, u_2 \in R$ with $u_1 \leq u_2$, we have

$$|f(t, u_1) - f(t, u_2| \leq 8\varphi(|u_1 - u_2|),$$

Where $\varphi \in \Psi$.

Let $C([0,1])$ be the space of all continuous functions defined on [0, 1]. It is known that this space equipped with the cone metric $d(u_{1,} u_2) = \|u_1 - u_2\|_\infty = \max_{t\in[0,1]}|u_1(t) - u_2(t)|$ is a comp0le cone metric space.

We define the green function associated to (3.20) by

$$G(g_{1,} g_2) = \begin{cases} g_1, (1 - g_2), 0 \leq g_1 \leq g_2 \leq 1 \\ g_2(1 - g_1), 0 \leq g_2 \leq g_1 \leq 1. \end{cases}$$

Assume that there exist$u_0 \in ([0,1])$ such that

(ii) $u_0(t) \leq \int_0^t G(g_1, g_2) f\,(g_2, u_0(g_2) dg_{2.}$

If (i) and (ii) are satisfied, then the problem (3.20) has a unique solution $u \in C([0,1]$.

Proof: It is easy to see that $u \in C([0,1]$ is a unique solution of (3.20) if and only if u us a unique fixed point of the mapping $T: C([0,1] - C[0,1])$ defined by

$$T(x) = \int_0^t G(g_1, g_2) f\,(g_2, u_0(g_2) dg_{2.} \text{ for all } t \in [0,1].$$

If we define the function $\alpha: C([0,1] \times [0,1] \to [0, +\infty[$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } y(t) \geq x(t), \\ 0, & \text{Otherwise} \end{cases}$$

Thus, T is generalized α-φ mapping, since for all $u_{1,}\ u_2 \in ([0,1])$,we have

$$\begin{aligned} \alpha(u_1, u_2)\|T(u_1) - T(u_2)\|_\infty &\leq \varphi(\|u_1 - u_2\|_\infty) \\ &\leq \varphi[M((u_1, u_2)] \end{aligned}$$

See theorem 4.1 in [2.4]. Moreover, the condition (*) is satisfied by taking $z = min\{x, y\}, x, y \in C([0,1])$. But applying theorem 3.4, we get the desired result.

CONCLUSION

In this attempt, we establish and extension of new generalize α-φ contractive mapping in complete cone metric spaces and obtain fixed point. These results generalize, improve and extend the theorem [30]. Also give the example for verified the results with an application to the two boundary value problem of second order differential equations and is given here to illustrate the usability of the getting results.

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