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COMMON FIXED POINT THEOREMS OF GERAGHTY TYPE CONTRACTION MAPS IN RECTANGULAR METRIC SPACES

P. HARIKRISHNA*[1], R. SANTHI KUMAR[2]

[1]Department of Mathematics,
Centurion University of Technology and Management, Andhra Pradesh, India.

[2]Department of Mathematics,
AITAM College of Engineering, Tekkali, Andhra Pradesh, India.

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ABSTRACT

In this paper, we define generalized Geraghty contraction maps in rectangular metric spaces with an altering distance function for a pair of maps, and prove the existence of common fixed points. Our results extend the some of the known results and provided example in support of main theorem.

Keywords: *Common fixed point; rectangular metric spaces; Geraghty contraction; weakly compatible.*

AMS (2010) Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is very important tool in Non linear analysis. Banach contraction principle is one of the fundamental results in fixed point theory. There are several generalizations of metric spaces. In 1973, Geraghty [10] extended the Banach contraction theorem by replacing the contraction by a function with specific properties and proved the existence of fixed points, In 1984 Khan, Swalwh and Sessa [14] studied fixed points with altering distance functions. In 2000 Braniciari [4] generalized metric spaces, in which triangular inequality is replaced by quadrilateral inequality which is known as rectangular metric spaces. In such extensions some of the authors are focused on rectangular metric spaces and proved the existence of fixed and common fixed points. We refer [2, 4, 6, 13, and 15].

Definition 1.1: [4] Let X be a nonempty set.A function d: X x X → [0, ∞) satisfy the following conditions for all $x, y \in$ X and all distinct $u, v \in$ X each of them different from x and y

(i) $d\ (x, y) = 0$ if and only if $x = y$
(ii) $d\ (x, y) = d\ (y, x)$, and
(iii) $d\ (x, y) \leq d\ (x, u) + d\ (u, v) + d\ (v, y)$. (rectangular inequality)

Then the function d is called a rectangular (generalized) metric and the pair (X,d) is called a rectangular (generalized) metric space (in short RMS).

Definition 1.2: [4] Let (X, d) be a rectangular metric space (in short RMS) and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is called (RMS) convergent to x in X if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(ii) $\{x_n\}$ is called (RMS) Cauchy sequence if and only if for every $\epsilon > 0$ there exists positive integer N(ϵ) such that $d(x_n, x_m) < \epsilon$ for all m > n ≥ N(ϵ).

A rectangular metric space (X, d) is called complete if every (RMS) Cauchy sequence is a (g.m.s) convergent.

Definition 1.3: ([14]) A function ψ: $R^+ \to R^+$, R^+ = [0, ∞) is said to be an *altering distance function* if the following conditions hold:

(i) Ψ is continuous,
(ii) ψ is non-decreasing, and
(iii) ψ (t) = 0 if and only if t = 0.

Corresponding Author: P. HariKrishna*[1],
[1]Department of Mathematics,
Centurion University of Technology and Management, Andhra Pradesh, India.

In 1973, [10] Geraghty extended Banach contraction theorem by replacing the contraction constant by a function with specific properties.

$S = \{\beta: [0, \infty) \to [0, 1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

Definition 1.4: [10, 16] Let (X, d) be a metric space. A self map $f: X \to X$ *is* said to be a *Geraghty contraction* if there exists $\beta \in S$ such that $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ *for all* $x, y \in X$.

Theorem 1.5: [10] Let (X, d) be a complete metric space. Let $f: X \to X$ be a *Geraghty contraction*.
Then for any choice of initial point $x_0 \in X$, the iteration $\{x_n\}$ defined by $x_n = f(x_{n-1})$ for $n = 1, 2, 3, \ldots$ converges to the unique fixed point z of f in X.

In 2019 P.H. Krishna [15] *et.al* proved the following fixed theorem in rectangular maps with admissible maps of Geraghty type contraction condition.

Definition 1.6: [15] Let (X, d) be a rectangular metric space and let $T: X \to X$ be a self map. If there exists $\beta \in S$ such that $d(Tx, Ty) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$
Where $M(x, y) = \max\{d(,y),\ d(x,Tx),\ d(y,Ty),\ \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty),\ \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty)\}$ for all $x, y \in X$ then we call T is a φ_M - generalized Geraghty contraction in rectangular metric spaces.

Theorem 1.7: [15] Let (X, d) be a Hausdorff and complete rectangular metric space.
Let $T: X \to X$ be an α- admissible mapping with respect to η. Assume that there exists an altering distance function φ such that $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$, implies $d(x,y) \leq \beta(\varphi(M(x,y)))\varphi(M(x,y))$ (2.1.1)
where $M(x, y) = \max\{ d(,y), d(x,Tx), d(y,Ty), \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty), \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty) \}$
Also, suppose that the following assertions are hold;

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$
(ii) for all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $\alpha(x, z) \geq \eta(x, z)$
(iii) T is continuous.

Then T has a periodic point $a \in X$ and $\alpha(x, Ta) \geq \eta(a, Ta)$ holds for each periodic point then T has a fixed point.

Definition 1.8: [12] Let f and g be self-mappings of a nonempty set X.
A point x in X is said to be common fixed point of f and g if x = fx = gx.
A point x in X is said to be coincidence point of f and g if fx = gx. And if u = fx = gx, then u is said to be a point of coincidence of f and g.
The mappings f, g $X \to X$ are said to be weakly compatible if they commute at their coincidence points. i.e., fgx= gfx whenever fx=gx.

Definition 1.9: [12] Two self mappings S and T of a rectangular metric space (X, d) are said to be *compatible* if $d(S(T(x_n), T(S(x_n))) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = u$ for some $u \in X$.

Definition 1.10: Suppose that (X, d, k) be a rectangular metric spaces and S, T: $X \to X$ be a map defined on X. Then S and T are said to be weakly commutative iff $d(S(T(x)), T(S(x))) \leq d(S(x), T(x))$ for all x in X. 2013, Muhammad Arsad *et.al* [13] proved common fixed point theorem on Hausdorff rectangular metric spaces.

Theorem 1.11: [13] Let (X, d) be a Hausdorff rectangular metric space and let F, g: $X \to X$ be self mappings such that $Fx \subset g(X)$. Asuume that (gX,d) is a complete rectangular metric space. Suppose that the following conditions hold.

$\psi(d(F(x), F(y)) \leq (\psi(M(g(x), g(y)) - \Phi(M(g(x), g(y)))$ for all x, y in X and $\psi, \Phi\ in\ \psi$,

where ψ is non decreasing and $M(g(x), g(y) = \max\{d(g(x), g(y),\ d(g(x), f(x),\ d(g(y), F(y))\}$. Then F and g have a unique coincident point in X. Moreover, if F and g are weakly compatible, then F and g have a unique common fixed point.

The following Lemma is useful to prove the Cauchy's sequence.
Lemma 1.12: [7] Let (X, d) be metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. For each $k > 0$, corresponding to m(k), we can choose n(k) to be the smallest integer such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)\ 1}) < \epsilon$. It can be shown that the following identities are satisfied.

(i) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$
(ii) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$,
(iii) $\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon$, and (*iv*) $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$.

Now, we prove the existence of common fixed points of generalized Geraghty contraction maps with rectangular metric spaces for a pair of maps.

MAIN RESULTS

Now we define Geraghty contraction for a pair of maps in Hausdorff rectangular metric spaces.

Definition 2.1: Let (X, d) be a Hausdorff rectangular metric space. Let T and S be self maps on X. If there exists $\beta \in S$ such that

$$d\big(S(x), S(y)\big) \leq \beta\big(\, d(T(x), T(y)\big)(d(T(x), T(y)) \tag{2.1.1}$$

for all $x, y \in X$, then we say that (T,S) is a pair of Geraghty contraction maps in Haussdorff rectangular metric spaces.

Definition 2.2: Let (X, d) be a Hausdorff rectangular metric space. Let T and S be self maps on X. If there exists $\beta \in S$ such that,

$$d\big(S(x), S(y)\big) \leq \beta(\, m(T(x), T(y)))(m(T(x), T(y))) \tag{2.2.1}$$

Where $m(T(x), T(y)) = \max\left\{ d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), \left[\frac{d(Tx,Sy)+d(Ty,Sx)}{2}\right]\right\}$ for all $x, y \in X$, then we say that (T,S) is a pair of Generalized Geraghty contraction maps in Haussdorff rectangular metric spaces.

Definition 2.3: Let (X, d) be a Hausdorff rectangular metric space. Let T and S be self maps on X. If there exists $\beta \in S$, $\psi \in \psi$ such that

$$\psi(d(S(x), S(y))) \leq \beta(\psi(\, m(T(x), T(y)))\psi(m(T(x), T(y))) \tag{2.3.1}$$

Where $m(T(x), T(y)) = \max\left\{ d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), \left[\frac{d(Tx,Sy)+d(Ty,Sx)}{2}\right]\right\}$ for all $x, y \in X$, then we say that (T,S) is a pair of ψ -Generalized Geraghty contraction maps in Haussdorff rectangular metric spaces.

Now we prove the existence of common fixed points of Geraghty contraction maps in pair of maps in ψ -Generalized Geraghty contraction maps in Haussdorff rectangular metric spaces.

Theorem 2.4: Let (X, d) be a Haussdorff rectangular metric space and let S, T: X→X be selfmaps such that S(X)⊂ $T(X)$. Assume that (Tx, d) be a complete rectangular metric space. Suppose that (T, S) is a pair of ψ -Generalized Geraghty contraction maps. Then S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof: Let $x_0 \in (X, d)$,

Since S(X) ⊂ $T(X)$, we define the sequence $Tx_n = S(x_{n-1})$ for each $n \geq 1$.

If $Tx_{n+1} = Tx_{n+2}$ for some n, then $Tx_{n+1} = Sx_{n+1}$ and hence $x_{n+1}$is a coincident point of $T\ and\ S.$ Without loss of generality, we assume that if $Tx_{n+1} \neq Tx_{n+2}$ for each n, then we have $d(Tx_{n+2}, Tx_{n+1}) > 0$

We consider

$$\Psi(d(Tx_{n+2}, Tx_{n+1})) = \psi(d(Sx_{n+1}, Sx_n)) \leq \beta(\psi(m(Tx_{n+1}, Tx_n)))\psi(m(Tx_{n+1}, Tx_n))\,) \tag{2.4.1}$$

$$\begin{aligned} m(Tx_{n+1}, Tx_n)) &= \max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Sx_{n+1}), d(Tx_n, Sx_n), [\tfrac{d(Tx_{n+1},Sx_n)+d(Tx_n,Sx_{n+1})}{2}]\} \\ &= \max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2}), d(Tx_n, Tx_{n+1}), [\frac{d(Tx_{n+1}, Tx_{n+1}) + d(Tx_n, Tx_{n+2})}{2}]\} \\ &= \max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2}), \tfrac{d(Tx_n,Tx_{n+2})}{2}\} \\ &\leq \max\left\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2}), \frac{d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})}{2}\right\} \\ &\leq \max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2})\}. \end{aligned}$$

If $\max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2})\} = d(Tx_{n+1}, Tx_{n+2})$

Since $\beta \in S$ then 2.4.1 implies

$\Psi(d(Tx_{n+2}, Tx_{n+1})) < \Psi(d(Tx_{n+2}, Tx_{n+1}))$, Which is a contradiction.

Therefore $\max\{ d(Tx_{n+1}, Tx_n), d(Tx_{n+1}, Tx_{n+2})\} = d(Tx_{n+1}, Tx_n)$

Now $\Psi(d(Tx_{n+2}, Tx_{n+1})) = \psi(d(Sx_{n+1}, Sx_n)) \leq \beta(\psi(m(Tx_{n+1}, Tx_n)))\psi(d(Tx_{n+1}, Tx_n)))$

Since $\beta \in S$,

$$\psi\big(d(Tx_{n+2},\ Tx_{n+1})\big) < \psi(d(Tx_{n+1},\ Tx_n)\ ) \tag{2.4.2}$$

Which follows that $\{\ \psi(d(Tx_{n+2},\ Tx_{n+1}))\}$ is a decreasing sequence of non-negative reals and since ψ is continuous, it folloows that $\{d(Tx_{n+2},\ Tx_{n+1})\}$ is also a decreasing sequence of non-negative reals so that $\lim_{n\to\infty} d(Tx_{n+2},\ Tx_{n+1})$ exists and it is r (say).

Now we show that r = 0.

If r > 0 then from (2.4.1) we have

$$\psi\big(d(Tx_{n+2},\ Tx_{n+1})\big) = \ \psi\big(d(Sx_{n+1},\ Sx_n)\big) \le \beta(\psi(\big(m(Tx_{n+1},Tx_n)\big))\psi(m(Tx_{n+1},\ Tx_n))$$

$$\frac{\psi(d(Tx_{n+2},Tx_{n+1}))}{\psi(d(Tx_{n+1},Tx_n))} \le \beta(\psi(m(Tx_{n+1},\ Tx_n))) < 1 \text{ for each } n \ge 1.$$

On letting $n \to \infty$, we get

$$1 = \ \lim_{n\to\infty} \frac{\psi(d(Tx_{n+2},Tx_{n+1}))}{\psi(d(Tx_{n+1},Tx_n))} \le \lim_{n\to\infty} \beta(\psi(\ m(Tx_{n+1},\ Tx_n))) \le 1.$$

So that $\beta(\psi(m(Tx_{n+1},\ Tx_n)))\ \to 1$ as $n \to \infty$, that implies

$$\lim_{n\to\infty} \psi(d(Tx_{n+1},\ Tx_n)\ ) = 0. \tag{2.4.3}$$

Hence r = 0.

Let $\{x_n\}$ be a sequence in X such that $d(Tx_{n+1},\ Tx_n) \to 0$ as $n \to \infty$.

Suppose that $\{Tx_n\}$ not a Cauchy sequence. Then there exists $\epsilon > 0$ integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that $d\big(Tx_{m(k)},\ Tx_{n(k)}\big) \ge \epsilon$ (2.4.4)

We choose $m(k)$, the least positive integer satisfying $d\big(Tx_{m(k)},\ Tx_{n(k)}\big) \ge \epsilon$, then

we have $m(k) > n(k) > k$ with $d\big(Tx_{m(k)},\ Tx_{n(k)}\big) \ge \epsilon, d\big(Tx_{m(k)-1},\ Tx_{n(k)}\big) < \epsilon$

$$\begin{aligned}\epsilon\ &\le d\big(Tx_{m(k)},T\,x_{n(k)}\big) \le d\big(Tx_{m(k)},T\,x_{m(k)-1}\big) + d\big(Tx_{m(k)-1},Tx_{n(k)}\big)\\ &< d\big(Tx_{m(k)},\ Tx_{m(k)-1}\big) +\ \epsilon.\end{aligned}$$

Since $d\big(Tx_{n(k)},Tx_{n(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \le d\big(Tx_{m(k)},\ Tx_{n(k)}\big) < \epsilon$.

This implies $d\big(Tx_{m(k)},\ Tx_{n(k)}\big) \to \epsilon$ as k → ∞.

$$\begin{aligned}\epsilon\ &\le d\big(Tx_{m(k)},Tx_{n(k)}\big)d\big(Tx_{m(k)},Tx_{m(k)-1}\big) +\ d\big(Tx_{m(k)-1},Tx_{n(k)+1}\big) + d\big(Tx_{n(k)+1},Tx_{n(k)}\big)\\ &< d\big(Tx_{m(k)},\ Tx_{m(k)-1}\big) + \epsilon + d\big(Tx_{n(k)+1},\ Tx_{n(k)}\big).\end{aligned}$$

Since $d\big(Tx_{n(k)},Tx_{n(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \le d\big(Tx_{m(k)},\ Tx_{n(k)}\big) < \epsilon$

This implies $d\big(Tx_{m(k)-1},\ Tx_{n(k)+1}\big) \to \epsilon$ as k → ∞.

$$\begin{aligned}\epsilon\ &\le d\big(Tx_{m(k)},T\,x_{n(k)}\big) \le d\big(Tx_{m(k)},Tx_{m(k)+1}\big) + d\big(Tx_{m(k)+1},Tx_{n(k)-1}\big) + d\big(Tx_{n(k)-1},Tx_{n(k)}\big)\\ &< d\big(Tx_{m(k)},\ Tx_{m(k)+1}\big) + \epsilon + d\big(Tx_{n(k)+1},\ Tx_{n(k)}\big).\end{aligned}$$

Since $d\big(Tx_{m(k)},Tx_{m(k)+1}\big) \to 0$ as $k \to \infty$, we have $\epsilon \le d\big(Tx_{m(k)},\ Tx_{n(k)}\big) < \epsilon$

This implies $d\big(Tx_{m(k)+1},\ Tx_{n(k)-1}\big) \to \epsilon$ as k → ∞.

Now, we consider

$$\begin{aligned}\psi(d(Tx_{m(k)+1}\ ,Tx_{n(k)})\ ) &= \psi(d(S(x_{m(k)}\ ), S(x_{n(k)-1})))\\ &\le \beta\left(\psi\left(m\big(Tx_{m(k)},Tx_{n(k)-1}\big)\right)\right)\psi\left(m\big(Tx_{m(k)},Tx_{n(k)-1}\big)\right)\end{aligned} \tag{2.4.5}$$

$$\begin{aligned}m\big(Tx_{m(k)},Tx_{n(k)-1}\big) &= \max\{d\big(Tx_{m(k)},T\,x_{n(k)}\big),\ \big(Tx_{m(k)},S\,x_{m(k)}\big),\ d\big(Tx_{n(k)},S\,x_{n(k)}\big),\\ &\qquad 1/2[d\big(Tx_{m(k)},Sx_{n(k)}\big) + d\big(Tx_{n(k)},S\,x_{m(k)}\big)]\}\\ &= \max\{d\big(Tx_{m(k)},T\,x_{n(k)}\big),\ \big(Tx_{m(k)},Tx_{m(k)+1}\big),\ d\big(Tx_{n(k)},Tx_{n(k)+1}\big),\\ &\qquad 1/2[d\big(Tx_{m(k)},Tx_{n(k)+1}\big) + d\big(Tx_{n(k)},T\,x_{m(k)} + 1\big)]\}\end{aligned}$$

On letting $k \to \infty$, and by Lemma (1.12), it follows that

$$\lim_{k\to\infty} m\left(Tx_{m(k)}, Tx_{n(k)-1}\right) = \max\{\epsilon, 0, 0, 1/2[\epsilon+\epsilon]\} = \epsilon.$$

$$\frac{d(Tx_{m(k)+1}\ , Tx_{n(k)})}{d(Tx_{m(k)}, Tx_{n(k)-1})} \leq \beta\left(\, d\left(Tx_{m(k)}, Tx_{n(k)-1}\right)\right) < 1$$

Therefore $\psi(\epsilon) \leq \lim_{k\to\infty} \beta\left(\, \psi(m\left(Tx_{m(k)}, Tx_{n(k)-1}\right)\right).\psi(\epsilon)$

$$1 = \frac{\psi(\epsilon)}{\psi(\epsilon)} \leq \lim_{k\to\infty} \beta\left(\psi(m\left(Tx_{m(k)}, Tx_{n(k)-1}\right)\right) \leq 1.$$

So that $\beta\left(\, \psi(m\left(Tx_{m(k)}, Tx_{n(k)-1}\right)\right) \to 1$ as $k \to \infty$,

Since $\beta \in S$, $\psi(m\left(Tx_{m(k)}, Tx_{n(k)-1}\right)) \to 0$ as $k \to \infty$.

Since ψ is continous, Hence it follows that $\epsilon = 0$, a contradiction.

Therefore $\{Tx_n\}$ is a Cauchy sequence in X, and since (TX, d) is a complete Haussdorff rectangular metric space, there exists $u \in$ Tx such that $Tx_{n+1} = Sx_n \to u$ as $n \to \infty$.

Now let $y \in X$ such that Ty= u .

Suppose that if $Sy \neq Ty$ i.e., $d(Sy, Ty) > 0$.

Since $\{Tx_n\}$ is a decreasing sequence of non-negative reals and $\{Tx_n\}$converges to Tu for some $u \in X$.

From the inequality (2.4.1), with $x = x_n$, we get

$$\psi(d(Sx_n\ , Sy)\,) \leq \beta\left(\psi\left(\, m(Tx_n, Ty)\right)\right)\psi(m(Tx_n, Ty)) \tag{2.4.6}$$

where

$$m(Tx_n, Ty) = \max\{\, (Tx_n, Ty), (Tx_n, Sx_n), (Ty, Sy), 1/2[(Tx_n, Sy) + (Ty, Sx_n)]$$
$$= \max\{\, (Tx_n, Ty), (Tx_n, Tx_{n+1}), (Ty, Sy), 1/2[(Tx_n, Sy) + (Ty, Tx_{n+1})]$$

On letting $n \to \infty$

= max{(u, Ty), (u,u), (Ty,Sy), ½[(Ty,Sy)+(Ty,u)]}
= max{(u, u), (u,u), (Ty,Sy), ½[(Ty,Sy)+(u,u)]}
= d(Ty, Sy)

Taking limit as $n \to \infty$ in (2.4.6)

$$\lim n \to \infty\ \psi(d(Sx_n\ , Sy)\,) \leq \lim n \to \infty\ \beta\left(\psi\left(\, m(Tx_n, Ty)\right)\right)\psi(m(Tx_n, Ty))$$
$$\lim n \to \infty\ \psi(d(Sx_n\ , Sy)\,) \leq \lim n \to \infty\ \beta\left(\psi\left(\, m(Tx_n, Ty)\right)\right)\psi(d(Ty, Sy))$$
$$\lim n \to \infty\ d(Sx_n, Sy) = 0$$
$$d(Ty, Sy) = 0$$

Therefore Ty=Sy and hence y is a coincidence point of T and S.

Since T(X) is complete, there exists $u \in T(X)$ such that $\lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Sx_n = Tu = u$ for some $y \in X$.

Suppose that if $Sy \neq Ty$ i.e., $d(Sy, Ty) > 0$

We get Since S and are weakly commutative

$$\psi(d(T(S(x_n), S(T(x_n))) \leq \psi(d(T(x_n), S(x_n))$$

Which implies ψ (d(T(u), S(u))) $\leq \psi(d(u, u))$

Therefore T(u) = S(u) and hence u is a coincidence point of S and T.

So that T(S(u)) = S(T(u))= S(S(u))

Suppose that $\psi(d(S(u), S(S(u))) > 0$

$$\psi(d(S(u), S\left(S(u)\right)) \leq \beta(\,\psi(m(T(u), T(S(u)))\psi(m(T(u), T\left(S(u)\right)))$$

Since β *in* S, it follows that

$$\psi(\mathrm{d}(S(u), S(S(u))) < \psi(d(T(u), T\left(S(u)\right)))$$
$$\psi(d(S(u), S\left(S(u)\right)) < \psi(\, d(S(u), S\left(S(u)\right))),$$ which is contradiction

so that $\psi(d(S(u), S(S(u))\,)) = 0$

Since
ψ is continous , it follows that $d(S(u), S(S(u)) = 0 \; and \; hence \; S(u) = S(S(u) \; so \; that \; S(u) is \; a \; fixed \; point \; of \; S.$ $S(u)$ is common fixed point of S and T.

Theorem 2.5: Let (X, d) be a Haussdirff rectangular metric space and let S, T: X→X be selfmaps such that S(X)⊂ $T(X)$. Assume that (Tx, d) be a complete rectangular metric space. Suppose that (T, S) is a pair of ψ -Generalized Geraghty contraction maps. Then S and T have a unique coincidence point in X. Moreover, if S and T commute, then S and T have a unique common fixed point.

Proof: As in the proof of the theorem {T(x_n)} is a Cauchy sequence in X, and since (X,d) is a Haussdirff rectangular metric space,{S(x_n) converges in (X,d) . Since T(X) is complete, there exists $u \in T\,X$ such that

lim$n$→∞ {T(x_{n+1})} = lim n →∞ {S(x_n)} = $T\,u = u$ for some $y \in X$. Suppose that if $Sy \neq Ty$

i.e., $d(Sy, Ty\ )$ >0
Since S and T commutes so that S (T(x_n)= T (S(x_n)) for all n.
Thus S(u) = T(u), and consequently by commutativity, T(T(u)) = T(S(u)) = S(S(u)).
So that T(S(u)) = S(T(u)) = S(S(u)).

Suppose that $d(S(u), S(S(u))$ > 0

$$d(S(u), S(S(u)) \leq \beta(\, d(T(u), T(S(u))d(T(u), T(S(u)))$$

$Since \; \beta \; in \; S,$ it follows that

$$\psi(d(S(u), S(S(u))) < \psi(d(T(u), T(S(u)))$$

$$\psi(d(S(u), S(S(u))) < \psi(\, d(S(u), S(S(u))))\text{, which is contradiction}$$

$so \; that \; \psi(d(S(u), S(S(u))) \; = \; 0 \; \; and \; hence \; \; S(u)) = S(S(u) \; so \; thas \; S(u) is \; a \; fixed \; point \; of \; S.$ S(u) is common fixed point of S and T.
The following is the Example in support of Theorem 2.4.

Example 2.5: Let X= A U B, where A=$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$ and B= [2, 3]. We define the generalized metric d on X such that

$$d\left(\frac{1}{2}, \frac{2}{3}\right) = 0.2,\ d\left(\frac{1}{2}, \frac{3}{4}\right) = 0.6,\ d\left(\frac{2}{3}, \frac{3}{4}\right) = 0.3,$$

$$d\left(\frac{1}{2}, \frac{1}{2}\right) = d\left(\frac{2}{3}, \frac{2}{3}\right) = d\left(\frac{3}{4}, \frac{3}{4}\right) = 0.$$

$d(\, x\, , y) = |x - y|$, if $x\, , y\ \in$ B or $x \in$ A.

We define S: X → X by $\begin{cases} \frac{2}{3} \; if \; x \; \in [2,3] \\ \frac{1}{2} \; \; if \; \; x \; \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\} \end{cases}$ and T: X → X by $\begin{cases} \frac{3}{4} \; if \; x \; \in [2,3] \\ \frac{1}{2} \; if \; x = \frac{1}{2} \\ \frac{2}{3} \; \; if \; \; x \; \in \left\{\frac{2}{3}, \frac{3}{4}\right\} \end{cases}$

We define β : [0, ∞) → [0, 1) by $\beta(t) = \begin{cases} 0 \; if \; t = 0 \\ \frac{1}{1+t} \; \; if \; t > 0 \end{cases}$

And $\psi(t) = t^2$ for all t ≥ 0.

Now we verify the inequality (2.3.10) in the following cases:

Case-(i):$x \in [2,3] \; and \; y = \frac{1}{2}$

$$\psi(d(S(x), S(y))) = \psi(d\left(\frac{2}{3}, \frac{1}{2}\right)) = \psi(0.2) = 0.01$$

$$m(T(x), T(y) = \max\{d\left(\frac{3}{4}, \frac{1}{3}\right), d\left(\frac{3}{4}, \frac{2}{3}\right), d\left(\frac{1}{2}, \frac{1}{2}\right), [\frac{d\left(\frac{2}{3}, \frac{1}{2}\right) + d\left(\frac{3}{4}, \frac{1}{2}\right)}{2}]\}$$

$$= \max\{0.6,\ 0.3, 0, (0.2+0.6)/2\} = 0.6$$

$$\beta(\psi(\, 0.6))\psi(0.6)) = \beta(0.36)(0.36) = \frac{0.36}{1.36} = 0.264$$

Therefore, $0.04 = \psi(d(S(x), S(y))) \leq 0.264 = \beta(\psi(\, m(T(x), T(y)))\psi(m(T(x), T(y)))$

Case-(ii): $x \in [2,3] \; and \; y \in \left\{\frac{2}{3}, \frac{3}{4}\right\}$

$$\psi(d(S(x), S(y))) = \psi(d\left(\frac{2}{3}, \frac{1}{2}\right)) = \psi(0.2) = 0.04$$

$$m(T(x), T(y) = \max\{\, d\left(\frac{3}{4}, \frac{2}{3}\right), d\left(\frac{3}{4}, \frac{2}{3}\right), d\left(\frac{2}{3}, \frac{1}{2}\right), [\frac{d\left(\frac{2}{3}, \frac{2}{3}\right) + d\left(\frac{3}{4}, \frac{1}{2}\right)}{2}]\}$$

$$= \max\{0.3,\ 0.3, 0.2, (0+0.6)/2\} = 0.3$$

$$(\psi(\, 0.3)\psi(0.3)) = \beta(0.09)(0.09) = \frac{0.09}{1.09} = 0.0825$$

$$0.04 = \psi(d(S(x), S(y))) \leq 0.0825 = \beta(\psi(\, m(T(x), T(y)))\psi(m(T(x), T(y)))$$

And the remaining cases also the inequality (2.3.1) holds.
Therefore S and T have the unique common fixed point $\frac{1}{2}$.

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