International Research Journal of Pure Algebra-10(8), 2020, 26-31
Available online through www.rjpa.info ISSN 2248–9037

σ – PURITY AND σ- REGULAR RINGS AND MODULES

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(Received On: 02-07-2020; Revised & Accepted On: 31-07-2020)

ABSTRACT

The aim of this paper is to relativize the concept of $M-$ purity and σ - purity defined and studied by Azumaya [6] with respect to an arbitrary hereditary torsion theory given by a left exact torsion radical σ and also relates these concepts with the notions of σ $-$ purity as given by B. B. Bhattacharya and D. P. Choudhury [7]. We also develope the theory of $(M;\ \sigma)-$ purity and $(\mu,\sigma)-$ purity relative to a torsion theory with radical σ where M is a finitely generated or cyclic $R-$ module and $\mu=\ (r_{ij})$ is an $i\times j$ matrix determined by a system of linear equations $\sum r_{ij}\ x_j\ =\ y_i$ where $y_i\in Y$ (a left $R-$ module) for each $i\ \in\ I$ and $j\ \in\ J$ are unknowns, which is weaker than the usual purity and given a sufficient condition for these two coincide. In this present paper we relativize the concept of the $\sigma-$ pure and $\sigma-$ flatness of a module. We also discuss about $\sigma-$ regular modules and weakly $\sigma-$ regular modules and its inter relationship. We also discuss about finitely generated $\sigma-$ flat modules and its condition for $\sigma-$ projectivity in Noetherian ring.

Key words: *Left $R-$ modules,$M-$ purity, $(M,\sigma)-$ purity, $\sigma-$ pure modules, $(\mu,\sigma)-$ purity, $\sigma-$ flat modules, $\sigma-$ regular modules, weakly $\sigma-$ regular modules, $\sigma-$ projective modules.*

Subject classification: *16D99.*

1. INTRODUCTION

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In the first section of this paper we examine the purities by torsion modules, finitely generated torsion modules and cyclic torsion modules. Work in this direction was initiated by Walker [17], Stenstrom [14], Azumaya [6], B. B. Bhattacharya and D. P. Choudhury [7]. In this there is an attempt to relativize the usual Cohn [9] purity with respect to a torsion theory. We also develop the theory of $(M;\ \sigma)-$ purity and $(\mu,\sigma)-$ purity relative to a torsion theory with radical σ where M is a finitely generated or cyclic $R-$ module and $\mu=\ (r_{ij})$ is an $i\ \times\ j$ matrix determined by a system of linear equations $\sum r_{ij}\ x_j\ =\ y_i$ where $y_i\ \in\ Y$ (a left $R-$ module) for each $i\ \in\ I$ and $j\ \in\ J$ are unknowns, which is weaker than the usual purity and given a sufficient condition for these two coincide. A submodule A is $(\mu,\sigma)-$pure in an $R-$ module B if any system of linear equations $\sum r_{ij}\ x_j\ =\ a_i$ given by the row finite matrix $\mu=\ (r_{ij})$ in A. Whenever solvable in B in the form $x_j\ =\ b_j$ for which there are left ideals $D_i\in\ D$ (Where D is the Gabriel filter [16] of dense left ideals corresponding to the left exact torsion radical σ such that $D_j\,b_j\ \in\ A$). The system is also solvable in A that is there are $a_j^{'}\in A$, with $\sum r_{ij}\ a_j^{'}\ =\ a_i$ for each $i\ \in\ I$ and $j\ \in\ J$. We view $\mu\colon \prod_J B\ \longrightarrow\ \prod_I B$ as a mapping by left matrix multiplication. In this present paper we relativize the concept of the $\sigma-$ pure and $\sigma-$flatness of a module. We also discuss about $\sigma-$ regular modules and weakly $\sigma-$ regular modules and its inter relationship. We also discuss about finitely generated $\sigma-$ flat modules and its condition for $\sigma-$projectivity in Noetherian ring. In this paper σ will denote a given left exact torsion radical and a torsion module means a module M for which $\sigma(M)\ =\ M$. Suppose that M,B and C are left $R-$ modules.

Definition 1.1: An epimorphism $p:\ B\ \longrightarrow\ C$ is said to be $(M,\sigma)-$ pure if for each homomorphism $\emptyset:\ M\ \longrightarrow\ C$ with image $(\emptyset)$ a torsion module that is $\emptyset[M]\ \subseteq\ \sigma[B]$, there exists a homomorphism $\varphi:\ M\ \rightarrow\ B$ such that $po\varphi\ =\emptyset$

$$\begin{tikzcd} & M \arrow[d, "\varphi"'] & \\ B \arrow[r, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

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We may extend the lower sequence by taking the kernel of p denoted by A and refer the short exact sequence $0 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 0$ as $(M,\sigma) -$ pure. If the epimorphism $p : B \longrightarrow C$ factorizes as $A \underset{g}{\to} N \underset{h}{\to} C$ that is $p = (hog)$ with $g -$ epic, then we can easily see that whenever g and h are $(M,\sigma) -$ pure then p is (M, σ) - pure.

Conversely in the above situation, if p is $(M,\sigma) -$ pure then h is also $(M,\sigma) -$ pure. If B is torsion then $p : B \to C$ splits (that is kernel p is direct summand of A) if and only if p is $(B,\sigma) -$ pure and this is equivalent to the condition that p is $(M,\sigma) -$ pure for every left $R -$ module M. Given a row finite $I \times J$ matrix $\mu = (r_{ij})$, by a system of linear equations given by μ in a left module Y , we mean a system $\sum r_{ij}\, x_j = y_i$ where $y_i \in Y$ (a left $R -$ module) for each $i \in I$ and $j \in J$ are unknowns.

Definition 1.2: We say that a submodule A is (μ,σ) −pure in an $R -$ module B, if any system of linear equation $\sum r_{ij}\, x_j = a_i$ given by the row finite matrix μ in A, whenever solvable in B in the form $x_j = b_i$ for which there are left $ideals\ D_i \in D$ where D is the Gabriel filter[16] of dense left ideals corresponding to the left exact torsion radical σ, such that $D_j b_j \subseteq A$. The system is also solvable in A that is there are $a_j' \in A$, with $\sum r_{ij}\, a_j' = a_i$ for each $i \in I$ and $j \in J$. This exactly means that given vectors $(b_j) \in \prod_J B$ and $(a_i) \in \prod_I A$ and $\mu(b_j) = a_i$ with $D_j b_j \in A$ for some $D_i \in D$, there exists $(a_j{}') \in \prod_I A$ such that $\mu(a_j{}') = a_i$ where the vector $\mu(a_j{}')$ is obtained by matrix product of the row finite matrix μ and column vector $(a_j{}')$. We may rephrase the above condition that a submodule A is (μ,σ) −pure in an $R -$ module B or that B is a (μ,σ) −pure extension of A as follows.

We view μ as mapping $\prod_J B$ to $\prod_I B$ by left matrix multiplication. Then we have:

Theorem 1.3: A submodule $A$is (μ,σ) −pure in B if and only if $\mu[\prod_J B] \cap \prod_J A \subseteq \mu[\prod_J A]$ whenever B_j'are submodules of B containing A such that A is dense in B_j.

Proof: Any element of the left hand side is of the form $(a_i)_I = \mu((b_j)_J) = \sum r_{ij}\, b_j$ and A dense in B_j means $B_j|A$ is torsion and hence for each element $(b_j + A) \in B_j|A$, there exists $D_j \in D$ such that $D_j(b_j + A) = 0$ that is $D_j(b_j) \subseteq A$.

The following result links (μ,σ) −purity with (M, σ) −purity.

Proposition 1.4: Let $\mu = (R_{ij})$ be a row finite $I \times J$ matrix where I and J are arbitrary sets. Then a submodule A is (μ,σ) −pure in a module B if and only if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B|A \longrightarrow 0$ is (M,σ) −pure where

$$\oplus_I R \xrightarrow{\mu'} \oplus_J R \longrightarrow M \longrightarrow 0 \text{ is exact with } \mu' \text{ given by the matrix } \mu.$$

Definition 1.5: A submodule A is $\mathcal{T} -$ **pure** in an $R -$ module M if and only if given a torsion submodule C of $M|A$, there exists a submodule B of M such that $B \cong C$ and $A \cap B = 0$. $(\mathcal{T}, \mathcal{T}_1)$ denotes a hereditary torsion theory with the corresponding idempotent kernel functor σ.

Definition 1.6: A submodule A of an $R -$ module M is called $\mu -$ pure in M where $\mu = (x_{ij})$ if whenever the system of linear equations $\sum r_{ij}\, x_j = a_i$, $i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $D_j \in D$, the associated Gabriel filter for left dense ideals, is solvable in M, it is solvable in A.

Definition1.7: A submodule A of an $R -$ module M is called $\sigma -$ pure in M if whenever a finite system of linear equations in a finite number of variables $\sum r_{ij}\, x_j = a_i$, $i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $D_j \in D$, the associated Gabriel filter for left dense ideals, is solvable in M, it is solvable in A.

Proposition 1.8: A submodule A of an $R -$ module is $\sigma -$ pure in M if and only if A is (Cohn)− pure [9] in the closure of A in M.

Proposition 1.9: A submodule A of an $R -$ module is μ −pure in M if and only if A is $M -$ pure in the closure of A in M where $\oplus_I R \xrightarrow{\mu'} \oplus_J R \longrightarrow M \longrightarrow 0$ is exact.

Proof: The closure $\bar{A}$ of A is defined by $\bar{A}|A = \sigma(M|A)$. If A is μ −pure in $\bar{A}$, then by Azumaya [6], A is μ −pure in $\bar{A}$. Then the given a finite system of linear equations in a finite number of variables $\sum r_{ij}\, b_j = a_i$,; $i \in I$ where $a_i \in A$ with $D_j(m_j) \subseteq A$ for some $D_j \in D$, $m_j + A \in \sigma(M|A) = \bar{A}|A$. Hence, $m_j \in \bar{A}$. As A is pure in $\bar{A}$ there exists $a_j{}' \in A$ such that $\sum r_{ij}\, a_j' = a_i$, and the system is solvable in A.

Conversely, if the given a finite system of linear equations in a finite number of variables $\sum r_{ij}\, m_j = a_i,\ i \in I$ with $a_i \in A$ and $m_j \in \bar{A}$ then, $m_j + A \in \bar{A}|A = \sigma(M|A)$ there is $D_j \in D$, $D_j\left(m_j + A\right) = 0$ that is $D_j\left(b_j\right) \subseteq A$ and hence the system is solvable in A and so, A is pure in $\bar{A}$. Hence A is μ −pur in $\bar{A}$ by Azumaya[6] proposition (1).

Definition 1.10:

(1) An $R -$ module C is said to be $\sigma -$ flat if a submodule A is $\sigma -$ pure in an $R -$ module B whenever $C \cong B|A$.

(2) A submodule A of an $R -$ module B is said to be $(\mu, \sigma) -$ pure if and only if $A \subseteq \bar{A}$ is $\mu -$ pure.

Proposition 1.11: Every torsion free module is σ −flat and every torsion σ −flat module is flat. Also, every flat module is $\sigma -$ flat of course.

Proposition 1.12: Consider the following conditions for the exact sequence
$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$

(i) $\sigma\,(C)$ is flat module.

(ii) A is $\sigma -$ pure in B.

(iii) C is $\sigma -$ flat.

(iv) $K\bar{A} \cap A = KA$ for all finitely generated right ideal K of $R.$ then $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$ if $\bar{A}$ is a flat module.

Proof:
$\boldsymbol{(i)} \Rightarrow \boldsymbol{(iii)}$: Given any sequence $0 \longrightarrow L \longrightarrow M \longrightarrow C \longrightarrow 0$, we have

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "I_L"'] & \bar{L} \arrow[r] \arrow[d] & \sigma(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$\sigma\,(C)$ is at implies L is pure in $\bar{L}$ implies L is $\sigma -$ pure in M implies C is $\sigma -$ flat.

$\boldsymbol{(iii)} \Rightarrow \boldsymbol{(ii)}$: This follows from the definition.

$\boldsymbol{(ii)} \Rightarrow \boldsymbol{(iv)}$: Since A is pure in $\bar{A}$ so, we have the ideal condition that $K\bar{A} \cap A = KA$ $f$or all finitely generated right ideal K of R.

$\boldsymbol{(iv)} \Rightarrow \boldsymbol{(i)}$**:** If in the exact sequence $0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow \sigma(B|A) \longrightarrow 0$, $\bar{A}$ is flat and $K\bar{A} \cap A = KA$, then A is pure in $\bar{A}$. This togather with $\bar{A}$ is flat gives that $\sigma(B|A)$ is flat.

Proposition 1.13: An $R -$ module A is $\sigma -$ pure in B if and only if every map f from a finitely presented module F to $B|A$ for which $Im(f)$ is torsion, lifts to a map from F to B.

Proof: It is given that f is torsion, this implies that f factors through $\sigma\,(B|A)$. As A is $\sigma -$ pure in B, also A is pure in $\bar{A}$ and hence g factors through $\bar{A}$ and hence f factors through B.

$$\begin{tikzcd}
 & & & F \arrow[d, "g"'] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "I_A"'] & \bar{A} \arrow[r] \arrow[d, "q"'] & \sigma(B|A) \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\lambda"] & B|A \arrow[r] & 0
\end{tikzcd}$$

Conversely, to show that A is $\sigma -$ pure in B, for this it is sufficient to prove that A is pure in $\bar{A}$ (in B of course). Hence given $g : F \longrightarrow (\bar{A}|A) = \sigma(B|A)$, $hog : F \longrightarrow B|A$ with $image(hog)$ torsion and hence there is $\mu : F \longrightarrow B$ such that $(\lambda o \mu) = (hog)$. But $\bar{A}$ is the pullback of H and g and hence there exists $\mu^{'}: F \longrightarrow \bar{A}$ such that $\lambda' o \mu' = g$ and $qo\mu^{'} = \mu$. Thus A is pure in $\bar{A}$.

Proposition 1.14: An $R -$ module A is a submodule of B and $\bar{A}$ is the closure of A in B, then the following are equivalent for an $R -$ module M given by a row finite defined matrix μ;

(i) A is $(M, \sigma) -$ pure in B

(ii) A is $(\mu, \sigma) -$ pure in B for every finite matrix $\mu = (r_{ij})$, for every finitely presented module M.

(iii) A is $\mu -$ pure in $\bar{A}$.

(iv) A is $M -$ pure in $\bar{A}$.

Note: Proof follows from the preceding propositions.

Proposition 1.15: If $A \subseteq B \subseteq C$, tnen the following statements hold:
(i) If A is $M-$ pure in B and B is $(M,\sigma)-$ pure in C, then A is $(M,\sigma)-$pure in C.
(ii) If A is $(M,\sigma)-$pure in C, then A is $(M,\sigma)-$ pure in B.
(iii) If A is$M-$ pure in C and $B|A$ is $(M,\sigma)-$pure in $C|A$, then B is $(M,\sigma)-$pure in C.
(iv) If B is $(M,\sigma)-$ pure in C then $B|A$ is $(M,\sigma)-$ pure in $C|A$.

Note: Proof follows from the preceding propositions.

Proposition 1.16: Suppose that for an $R-$ module M, $(M|\ \sigma(M))$ is a flat module then the following are equivalent:
(i) M is $(\sigma)-$ flat
(ii) $\sigma(M))$ is flat.
(iii) M is flat.

Proof: We write M as $B|A$ with B projective. We have the commutative diagram with exact rows and columns

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "I_A"'] & \bar{A} \arrow[r] \arrow[d] & \sigma(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
& & B|\bar{A} \arrow[r, "I_d"] \arrow[d] & M|\sigma(M) \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$: If M is $(\sigma)-$ flat then A is $\sigma-$ pure in B that is A is pure in $\bar{A}$. Since $M|\sigma(M)$ and B are flat this implies $\bar{A}$ is flat. Now A is $\sigma-$ pure in B and $\bar{A}$ is flat this implies $\sigma(M)$is flat.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{iii})$: If $\sigma(M)$ is flat, M is flat because $M|\sigma(M)$ is flat.

$(\boldsymbol{iii}) \Longrightarrow (\boldsymbol{i})$: If M is flat, M is $(\sigma)-$ flat always.

The proof of the following these two propositions are analogous to the usual purity (Fieldhouse[12]).

Proposition 1.17: If $A \subseteq B \subseteq C$, then the following statements hold:
(i) If A is pure in B and B is $\sigma-$ pure in C, then A is $\sigma-$ pure in C.
(ii) If A is $\sigma-$ pure in C, then A is $\sigma-$ pure in B.
(iii) If A is $\sigma-$ pure in C and $B|A$ is pure in $C|A$, then B is $\sigma-$ pure in C.
(iv) If B is $\sigma-$ pure in C then $B|A$ is $\sigma-$ pure in $C|A$.

Proposition 1.18: An $R-$ module M is $\sigma-$ flat if and only if the given a finite system of linear equations in a finite number of variables $\sum r_i\, m_i = 0$, $i \in I$ with $m_i \in \sigma(M)$ and $r_i \in R$, there exists $y_i \in M$ and $r_{ij} \in R$ such that $m_i = \sum r_{ij}\, y_j$ and $\sum r_i\, r_{kj} = 0$.

Definition 1.19: A sequence $0 \longrightarrow A \longrightarrow B \longrightarrow (B|A) \longrightarrow 0$ is said to be **weakly** $(N,\ \sigma)-$ pure if the sequence $0 \longrightarrow N \otimes A \longrightarrow N \otimes \bar{A}$ is exact, where N is a given right $R-$ module and $\bar{A}$ is the closure of A in B, that is $\bar{A}|A = \sigma(B|A)$.

Now we consider a column finite matrix so that it arises as the defining matrix of a right $R-$ module.

Definition 1.20: Given a column finite matrix $\nu = (s_{ij})$, an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow (B|A) \longrightarrow 0$ is said to be **weakly** $(\nu,\ \sigma)-$ pure if given a system of equations$\sum s_{ij}\ x_j = a_i$ with $(x_j) \in \oplus_J B$, $(a_i) \in \oplus_I A$ such that for each $j \in J$, there is $D_j \in D$ with $D_j x_j \subseteq A$, there exists $(a_j') \in \oplus_J A$ with $\sum s_{ij}\ a_j' = a_i$.

Proposition 1.21: For an exact sequence $0\longrightarrow A \longrightarrow B \longrightarrow (B|A) \longrightarrow 0$ of left $R-$ modules and a column finite matrix $\nu = (s_{ij})$, the following statements are equivalent:

(i) The sequence is weakly $(\nu,\ \sigma)-$ pure.

(ii) The sequence is weakly $(N;\ \sigma)-$ pure for a right $R-$ module N given by a column finite matrix $\nu = \left(s_{ij}\right)$.

(iii) A is weakly $N-$ pure in $\bar{A}$ (where $\bar{A}|A = \sigma(B|A)$).

Proof: $(iii) \implies (ii)$ it follows from the definition of weakly $(N;\ \sigma)-$ purity By proposition(2), Azumaya [6]. A is weakly $N-$ pure in $\bar{A}$ if and only if A is weakly $\nu = (s_{ij})-$ pure in $\bar{A}$. Now the last condition means that given $\left(x_j\right) \in \oplus_J \bar{A}$ and $(a_i) \in \oplus_I A$ with $\sum r_{ij}\, x_j = a_i, i \in I$ there exists $(a_j{}') \in \oplus_J A$ with $\sum r_{ij}\, a'_j = a_i, i \in I$. But $x_j \in \bar{A}$ means that $(x_j + A) \in \sigma(B|A)$ that is $D_j x_j \subseteq A$ and hence A is weakly $\nu = (s_{ij})-$ pure in $\bar{A}$ if and only if A is weakly $(\nu,\ \sigma)-$ pure in B.
This proves the equivalence of (i) and (iii).

Definition 1.22: A left $R-$ module M is said to be $\sigma-$ regular if every submodule of M is $\sigma-$ pure. It is said to be weakly $\sigma-$ regular if every dense submodule of M is pure (in the sense of Cohn) [9].

Note: Every $\sigma-$ regular module is weakly $\sigma-$ regular.

Proposition 1.23: A direct sum of weakly $\sigma-$ regular modules is weakly $\sigma-$ regular.

Proof: Suppose M_1, M_2 be weakly $\sigma-$ regular modules. Let N be a dense sub-module of $M_1 \oplus M_2$. Let $p_1,\ p_2$ denotes the projections onto the summands. Then $p_2(N)$ is a dense submodule of M_2, for given $m_2 \in M_2$, there exists $D \in D$, the Gabriel filter such that $D(0, m_2) \subseteq N$, then $Dm_2 \subseteq p_2(N)$. Consider $\sum r_{ij}\, x_j = a_i, i \in I$, a finite system of euqations with $x_j \in M_1 \oplus M_2$ and $n_1 \in N$. Now $\sum r_{ij}\, p_2(x_j) = p_2(n_i), i \in I$. As $p_2(N)$ is pure in M_2, because M_2 is weakly $\sigma-$ regular and $p_2(N)$ is dense in M_2, there exists $y_j \in N$ such that $\sum r_{ij}\, p_2(y_j) = p_2(n_i), i \in I$. Hence, $\sum r_{ij}\, y_j - n_i = z_i \in M_1$. Hence, $z_i \in M_1 \cap N$ and $\sum r_{ij}\, (y_j - x_j) = z_i$. Taking projection onto

$$M_1. \sum r_{ij}\, (p_1(y_j) - p_1(x_j)) = p_1(z_i), i \in I.$$

So, we get a finite system of linear equations which is solvable in M_1 with $z_i \in M_1 \cap N$ and $M_1|M_1 \cap N \cong$ *M1+NN⊆M|N*. Hence *(M1∩N)* is dense in *M1*. So, there exits *qj∈(M1 ∩N)* such that

$$\sum r_{ij}\, q_j = z_i = \sum r_{ij}\, y_j - n_i .$$

Hence, $\sum r_{ij}\left(y_j - q_j\right) = n_i$ and $\left(y_j - q_j\right) \in N$ and so N is pure in M. Hence, finite direct sums of weakly $\sigma-$ regular modules are weakly $\sigma-$ regular. Now suppose that $(M_i)_{i\in I}$ is a family of weakly $\sigma-$ regular modules. Let $N \subseteq \oplus_i M_i$ be a dense submodule.

Let $\sum r_{ij}\, x_j = n_i$ be a finite system of equations with $x_j \in N \subseteq \oplus_i M_i$. Now all the x_j 's are contained in a finite direct sum $\oplus_i M_i$. $\oplus_i M_i|(\oplus_i M_i \cap N) \cong (N + \oplus_i M_i)|N \subseteq \oplus_i M_i|N \in \mathcal{T}$. Hence $((\oplus_i M_i) \cap N)$ is dense in $(\oplus_i M_i)$ and $n_i \in ((\oplus_i M_i) \cap N)$. By the above, $(\oplus_i M_i)$ is weakly $\sigma-$ regular modules and so the equations are solvable in $((\oplus_i M_i) \cap N)$ and hence in N. This proves that $(\oplus_i M_i)$ is weakly $\sigma-$ regular modules.

Proposition 1.24: Dense submodules and factors of weakly $\sigma-$ regular modules are again weakly $\sigma-$ regular modules.

Proof: Suppose that M be an $R-$ module which is weakly $\sigma-$regular. We consider the sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0 \text{ Let } L\text{'be dense in } L.$$

Now $L' = M'|N$ for some $M' \subseteq M$ and L|L' $= M|M' \in \mathcal{T}$. Then M' is pure in M and hence $M'|N$ is pure in $M|N$ that is L' is pure in L.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "I_N"'] & M' \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & M \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Let N be dense in M that is $M|N \in \mathcal{T}$. If N' be a dense submodule of N that is $N|N' \in \mathcal{T}$ then $M|N' \in \mathcal{T}$ as $0 \longrightarrow N|N' \longrightarrow M|N' \longrightarrow M|N \longrightarrow 0$ is exact. Thus M being weakly $\sigma-$ regular, N' is pure in M and so N' is pure in N also.

Remark: Now we give a generalization of a result of Fieldhouse [**12**].

Proposition 1.25: Let $0 \longrightarrow A \longrightarrow B \overset{\lambda}{\rightarrow} C \longrightarrow 0$ be exact. Then B is $\sigma -$ regular implies A and C are $\sigma -$ regular. Conversely, C is $\sigma -$ regular, A is regular, A is pure in B implies B is $\sigma -$ regular.

Proof: Suppose $\bar{A}$ that the closure of A in B. If $A' \subseteq A$, then A' is $\sigma -$ pure in B and hence $A^{'}$ is pure in $\bar{A}$ which is closure of A in B. But $\bar{A} \subseteq \bar{B}$ and hence $A^{'}$ is pure in $\bar{A}$ and so $A^{'}$ is $\sigma -$ pure in A. If $C^{'} \subseteq C$, and let $\overline{C}^{'}$ be a closure of $C^{'}$ in C. Let $P = \lambda^{-1}(C^{'})$, then $B|P \cong C|C^{'}$ and $|A = C^{'}$, $B|A \cong C$. Now by the previous proposition P is $\sigma -$ pure in B, this implies that C' is $\sigma -$ pure in C.

Conversely, if A is regular, C is $\sigma -$ regular and A is pure in B, then if $B' \subseteq B$, the conclusion follows trivially a system of linear equations $\sum r_{ij}\, b_j = b'_i$ with $D_j b_j \subseteq B'$, $D_j \in D$.

Proposition 1.26: The following conditions are equivalent on a ring R:

(i) Every torsion module is flat.
(ii) Every module is $\sigma -$ flat.
(iii) Every module is $\sigma -$ regular.
(iv) Every module is weakly $\sigma -$ regular.
(v) R is a $\sigma -$ regular module.
(vi) R is a weakly $\sigma -$ regular module.

Proposition 1.27: Let R be a left Noetherian ring, any finitely generated $\sigma -$ flat module is $\sigma -$ projective.

Proof: If M is finitely generated $\sigma -$ flat module, $\sigma(M)$ is a finitely presented torsion module and hence the inclusion map $\sigma\ (M) \implies M$ lifts. So, it is $\sigma -$projective.

Proposition 1.28: If M_i be a finite family of modules, each of whose dense submodules are finitely generated, then $(\bigoplus_i M_i)$ also has this property.

Proposition 1.29: If every element I of the Gabriel filter D is finitely generated, then a sequence is $\sigma -$ pure if and only if every finitely presented torsion module is projective relative to it.

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Source of Support: Nil, Conflict of interest: None Declared