

ON A PRODUCT SUMMABILITY OF AN INFINITE SERIES

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Let $\sum a_n$ be a given infinite series with partial sums s_n . Let u_n^α denote the nth Cesaro mean of order $\alpha > -1$ of the sequence (s_n) . The series $\sum a_n$ is summable

$|C, \alpha|_k, k \geq 1$, if

$$(1) \quad \sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty \quad (\text{Flett [1]}).$$

For $\alpha = 1$, $|C, \alpha|_k$ reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of positive real constants such that $P_n = \dots \rightarrow \infty$, as $n \rightarrow \infty$ ($P_{-1} = p_{-1} = 0$).

The (N, p) transform ϕ_n of (s_n) generated by (p_n) is defined by

$$(2) \quad \phi_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v.$$

The sequence - to - sequence transformation

$$\Phi_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (Φ_n) of (\bar{N}, p_n) transform of (s_n) generated by (p_n) . The series $\sum a_n$ is summable

$|R, p_n|_k, k \geq 1$, if

$$(3) \quad \sum_{n=1}^{\infty} n^{k-1} |\Phi_n - \Phi_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all n (resp. $k=1$), $|R, p_n|_k$ summability reduces to $|C, 1|_k$ (resp. $|R, p_n|_k$) summability.

The series $\sum a_n$ is said to be summable $|(\bar{N}, p_n)(N, q_n)| \dots (N, p_n) \dots (N, q_n) \dots$, when the (N, p) transform of the (N, q) transform of (s_n) is a sequence of bounded variation (see Das [2]).

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We give the following new definition:

Let (T_n) defines the sequence of the (\bar{N}, q_n) transform of the (\bar{N}, p_n) transform of (s_n) generated by the sequences (q_n) and (p_n) respectively. The series $\sum a_n$ is said to be summable $|(R, q_n)(R, p_n)|_k, k \geq 1$, if

$$(4) \quad \sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty.$$

We may assume through the paper that $Q_n = q_0 + \dots + q_n \rightarrow \infty, \text{ as } n \rightarrow \infty, q_n / Q_n \rightarrow 0, \text{ as } n \rightarrow \infty.$

2. NEW RESULTS:

We state and prove the following

Theorem: 2.1 Let $k \geq 1, (\lambda_n)$ be a sequence of constants. Define

$$f_v = \sum_{r=v}^n \frac{q_r}{P_r}, \quad F_v = \sum_{r=v}^n p_r f_r.$$

Let

$$(5) \quad p_v Q_v = O(P_v),$$

$$(6) \quad P_v f_v = O(vq_v),$$

$$(7) \quad \sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left(\frac{(vq_v)^{k-1}}{Q_v^k}\right).$$

Then sufficient conditions for the implication

$$(8) \quad \sum a_n \text{ is summable } |R, q_n|_k \Rightarrow \sum a_n \lambda_n \text{ is summable } |(R, q_n)(R, p_n)|_k$$

are

$$(9) \quad |\lambda_n| < Q_n,$$

$$(10) \quad vp_v |\lambda_v| = O(P_v),$$

$$(11) \quad |\lambda_v| F_v = O(Q_v),$$

$$(12) \quad |\Delta \lambda_v| F_v = O(q_v),$$

and

$$(13) \quad |\Delta \lambda_v| = O(q_v).$$

Proof: Let (S_n) be the sequence of partial sums of $\sum a_n \lambda_n$. Let v_n, V_n be the $(\bar{N}, q_n), (\bar{N}, q_n)(\bar{N}, p_n)$ transforms of the sequences $(s_n), (S_n)$ respectively. We write $t_n = v_n - v_{n-1}, T_n = V_n - V_{n-1}$. Therefore

$$(14) \quad t_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} a_v,$$

and

$$\begin{aligned} V_n &= \frac{1}{Q_n} \sum_{r=0}^n q_r \frac{1}{P_r} \sum_{v=0}^r p_v S_v \\ &= \frac{1}{Q_n} \sum_{v=0}^n p_v S_v \sum_{r=v}^n \frac{q_r}{P_r} \\ &= \frac{1}{Q_n} \sum_{v=0}^n p_v S_v f_v. \end{aligned}$$

$$\begin{aligned}
 T_n &= V_n - V_{n-1} \\
 &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^n p_r S_r f_r + \frac{P_n S_n f_n}{Q_{n-1}} \\
 &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^v p_r f_r \sum_{v=0}^r a_v \lambda_v + \frac{P_n q_n}{P_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \\
 (15) \quad &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \sum_{r=v}^n p_r f_r + \frac{P_n q_n}{P_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \\
 &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} a_v \frac{\lambda_v}{Q_{v-1}} \sum_{r=v}^n p_r f_r + \frac{P_n q_n}{P_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} a_v \frac{\lambda_v}{Q_{v-1}} \\
 &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^v Q_{r-1} a_r \right) \Delta_v \left(\frac{\lambda_v}{Q_{v-1}} \sum_{r=v}^n p_r f_r \right) + \left(\sum_{v=1}^n Q_{v-1} a_v \right) \frac{\lambda_n P_n f_n}{Q_{n-1}} \right) \\
 &\quad + \frac{P_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^v Q_{r-1} a_r \right) \Delta \left(\frac{\lambda_v}{Q_{v-1}} \right) + \left(\sum_{v=1}^n Q_{v-1} a_v \right) \frac{\lambda_n}{Q_{n-1}} \right) \\
 &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v F_v + \frac{Q_{v-1}}{q_v} p_v t_v \lambda_v f_v + \frac{Q_{v-1}}{q_v} t_v \Delta \lambda_v F_{v+1} \right) \right) + \frac{P_n}{Q_{n-1}} t_n \lambda_n f_n \\
 &\quad + \frac{P_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v + \frac{Q_{v-1}}{q_v} t_v \Delta \lambda_v \right) \right) + \frac{P_n Q_n}{P_n Q_{n-1}} t_n \lambda_n, \\
 &= \sum_{j=1}^7 T_{nj}.
 \end{aligned}$$

In order to complete the proof, it sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nj}|^k < \infty, \quad j = 1, 2, 3, 4, 5, 6, 7.$$

Applying Hölder's inequality,

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n1}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} t_v \lambda_v F_v \right|^k \\
 &\leq \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{1}{q_v^{k-1}} |t_v|^k |\lambda_v|^k F_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
 &= O(1) \sum_{v=1}^m \frac{1}{q_v^{k-1}} |t_v|^k |\lambda_v|^k F_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \frac{|\lambda_v|^k F_v^k}{Q_v^k} = O(1).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n2}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1} P_v}{q_v} t_v \lambda_v f_v \right|^k \\
 &\leq \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{Q_{v-1}^k P_v^k}{q_v^{2k-1}} |t_v|^k |\lambda_v|^k f_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1}
 \end{aligned}$$

$$\begin{aligned}
 &= O(1) \sum_{v=1}^m \frac{Q_{v-1}^k P_v^k}{q_v^{2k-1}} |t_v|^k |\lambda_v|^k f_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \frac{P_v^k}{q_v^k} |\lambda_v|^k f_v^k \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\lambda_v|^k \left(\frac{v P_v}{P_v} \right)^k = O(1).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n3}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1}}{q_v} t_v \Delta \lambda_v F_{v+1} \right|^k \\
 &\leq \sum_{n=1}^{m+1} n^{k-1} \frac{q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{Q_{v-1}^k}{q_v^{2k-1}} |t_v|^k |\Delta \lambda_v|^k F_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^k \\
 &= O(1) \sum_{v=1}^m \frac{Q_{v-1}^k}{q_v^{2k-1}} |t_v|^k |\Delta \lambda_v|^k F_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\Delta \lambda_v|^k F_v^k \frac{1}{q_v} = O(1).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=1}^m n^{k-1} |T_{n4}|^k &= \sum_{n=1}^m n^{k-1} \left| \frac{P_n}{Q_{n-1}} t_n \lambda_n f_n \right|^k \\
 &= O(1) \sum_{n=1}^m n^{k-1} |t_n|^k |\lambda_n|^k \left(\frac{q_n}{Q_{n-1}} \right)^k \left(\frac{np_n}{P_n} \right)^k,
 \end{aligned}$$

as $\frac{q_n}{Q_{n-1}} = \frac{q_n}{Q_n - q_n} = \frac{1}{\frac{Q_n}{q_n} - 1} \rightarrow 0$,

$$\sum_{n=1}^{\infty} n^{k-1} |T_{n4}|^k = O(1) \sum_{n=1}^m n^{k-1} |t_n|^k |\lambda_n|^k \left(\frac{np_n}{P_n} \right)^k = O(1)$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n5}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{P_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n-1} t_v \lambda_v \right|^k \\
 &\leq \sum_{n=1}^{m+1} n^{k-1} \frac{P_n^k q_n^k}{P_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
 &= O(1) \sum_{v=1}^m |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{n^{k-1} P_n^k q_n^k}{P_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\lambda_v|^k \frac{1}{Q_v^k} = O(1).
 \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{m+1} n^{k-1} |T_{n6}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{P_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1}}{q_v} t_v \Delta \lambda_v \right|^k \\ &\leq \sum_{n=2}^{m+1} n^{k-1} \frac{P_n^k q_n^k}{P_n^k Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1}^k}{q^{2k-1}} |t_v|^k |\Delta \lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{v=1}^m \frac{Q_{v-1}^k}{q_v^{2k-1}} |t_v|^k |\Delta \lambda_v|^k \sum_{n=v+1}^{m+1} n^{k-1} \frac{P_n^k q_n^k}{P_n^k Q_{n-1}} \\ &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\Delta \lambda_v|^k \frac{1}{q_v^k} = O(1). \end{aligned}$$

Finally

$$\begin{aligned} \sum_{n=1}^m n^{k-1} |T_{n7}|^k &= \sum_{n=1}^m n^{k-1} \left| \frac{p_n Q_n}{P_n Q_{n-1}} t_n \lambda_n \right|^k \\ &= O(1) \sum_{n=1}^m n^{k-1} |t_n|^k |\lambda_n|^k \left(\frac{P_n}{P_n} \right)^k = O(1). \end{aligned}$$

This completes the proof of the theorem.

Theorem: 2.2 Let (7) be satisfied and

$$(16) \quad P_v = O(p_v Q_v),$$

$$(17) \quad Q_n = O(nq_n).$$

Then necessary conditions for the implication (8) to be satisfied are

$$|\lambda_v| = O\left(\frac{Q_{v-1}}{1 + F_v}\right), \quad |t_v| = O\left(\frac{v^{1-1/k} q_v}{p_v f_v}\right), \quad |\Delta \lambda_v| = O\left(\frac{v^{1-1/k} q_v}{1 + F_{v+1}}\right).$$

Proof: For $k \geq 1$ define

$$\begin{aligned} A^* &= \{(a_j): \sum a_j \text{ is summable } |R, q_n|_k\}, \\ B^* &= \{(b_j): \sum b_j \lambda_j \text{ is summable } |(R, q_n)(R, p_n)|_k\}. \end{aligned}$$

From (15), we have

$$(18) \quad T_n = \sum_{v=1}^n \left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) a_v \lambda_v$$

With t_n and T_n as defined by (14) and (18), the spaces A^* and B^* are BK-spaces with norms defined by

$$(19) \quad \|c\|_1 = \left\{ |t_0|^k + \sum_{n=1}^{\infty} n^{k-1} |t_n|^k \right\}^{1/k},$$

$$(20) \quad \|c\|_2 = \left\{ |T_0|^k + \sum_{n=1}^{\infty} n^{k-1} |T_n|^k \right\}^{1/k}.$$

respectively. By the hypothesis of the theorem,

$$(21) \quad \|c\|_1 < \infty \Rightarrow \|c\|_2 < \infty.$$

The inclusion map $i : A^* \rightarrow B^*$ defined by $i(a) = a$ is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exist a constant $K > 0$ such that

$$(22) \quad \|c\|_2 \leq K \|c\|_1.$$

Let e_n denote the n th coordinate vector. From (14) and (18), with (a_n) defined by $a_n = e_n - e_{n+1}$, $n = v$, $a_n = 0$ otherwise, we have

$$t_n = \begin{cases} 0, & n < v \\ \frac{q_v}{Q_v}, & n = v \\ -\frac{q_n q_v}{Q_n Q_{n-1}}, & n > v. \end{cases}$$

and

$$T_n = \begin{cases} 0, & n < v \\ \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{P_v q_v}{P_v Q_{v-1}} \right) \lambda_v, & n = v \\ \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{P_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right), & n > v. \end{cases}$$

From (19) and (20), we have

$$(23) \quad \|c\|_1 = \left\{ v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n q_v}{Q_n Q_{n-1}} \right)^k \right\}^{1/k},$$

$$(24) \quad \|c\|_2 = \left\{ v^{k-1} \left| \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{P_v q_v}{P_v Q_{v-1}} \right) \lambda_v \right|^k + \sum_{n=v+1}^{\infty} n^{k-1} \left| \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{P_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right) \right|^k \right\}^{1/k}$$

Applying (22), we obtain

$$(25) \quad v^{k-1} \left| \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{P_v q_v}{P_v Q_{v-1}} \right) \lambda_v \right|^k + \sum_{n=v+1}^{\infty} n^{k-1} \left| \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{P_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right) \right|^k \\ = O(1) \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n q_v}{Q_n Q_{n-1}} \right)^k \right).$$

As the R.H.S of (25), by (7), is

$$= O(1) \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + \frac{q_v^k}{Q_v^{k-1}} \sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \right)$$

$$\begin{aligned}
 &= O(1) \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + \left(\frac{q_v}{Q_v} \right)^{k-1} v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right) \\
 &= O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),
 \end{aligned}$$

and the fact that each term of the L.H.S of (25) is $O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right)$, we obtain

$$v^{k-1} \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{P_v q_v}{P_v Q_{v-1}} \right)^k |\lambda_v|^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),$$

which implies by (16)

$$\left(\frac{q_v}{Q_v Q_{v-1}} \right)^k (1 + F_v)^k |\lambda_v|^k = O \left(\frac{q_v}{Q_v} \right)^k,$$

that is

$$|\lambda_v| = O \left(\frac{Q_{v-1}}{1 + F_v} \right).$$

Also, we have, by (25),

$$(26) \quad \sum_{n=v+1}^{\infty} n^{k-1} \left| \left(\frac{q_n P_v f_v}{Q_n Q_{n-1}} \right) \lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{P_n q_n}{P_n Q_{n-1}} \right) \Delta \lambda_v \right|^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right).$$

The above, via the linear independence of λ_v and $\Delta \lambda_v$, implies

$$\sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{P_n q_n}{P_n Q_{n-1}} \right)^k |\Delta \lambda_v|^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right)$$

$$|\Delta \lambda_v|^k (1 + F_{v+1})^k \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right). \quad (\text{by (16)})$$

As by (17), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty} \Delta \left(\frac{1}{Q_{n-1}^k} \right) = O(1) \sum_{n=v+1}^{\infty} \frac{|\Delta Q_{n-1}^k|}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} \frac{Q_{n-1}^{k-1} q_n}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k, \text{ then,}$$

$$|\Delta \lambda_v|^k (1 + F_{v+1})^k \frac{1}{Q_v^k} = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),$$

which implies

$$\Delta \lambda_v = O \left(\frac{v^{1-1/k} q_v}{1 + F_{v+1}} \right).$$

Also, by (26),

$$\sum_{n=v+1}^{\infty} n^{k-1} \left| \frac{q_n p_v f_v}{Q_n Q_{n-1}} \lambda_v \right|^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),$$

$$p_v^k f_v^k |\lambda_v|^k \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),$$

$$p_v^k f_v^k |\lambda_v|^k \frac{1}{Q_v^k} = O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right),$$

which implies

$$\lambda_v = O \left(\frac{v^{1-1/k} q_v}{p_v f_v} \right).$$

3. APPLICATIONS:

Corollary: 3.1 Let $k \geq 1$. Define

$$(27) \quad f_v = \sum_{r=v}^n \frac{q_r}{r}, \quad F_v = \sum_{r=v}^n f_r.$$

Let

$$(28) \quad Q_v = O(v),$$

$$(29) \quad f_v = O(q_v).$$

Then sufficient conditions for the implication

$$(30) \quad \sum a_n \text{ is summable } |R, q_n|_k \Rightarrow \sum a_n \lambda_n \text{ is summable } |(R, q_n)(C, 1)|_k \text{ are}$$

$$(31) \quad |\lambda_n| < Q_n,$$

$$(32) \quad |\lambda_v| f_v = O(1),$$

$$(33) \quad |\lambda_v| F_v = O(Q_v),$$

$$(34) \quad |\Delta \lambda_v| F_v = O(q_v),$$

$$(35) \quad |\Delta \lambda_v| = O(q_v).$$

Proof: Follows from theorem 1 by putting $p_n = 1$ for all n.

Corollary: 3.2 Let $k \geq 1$. Define

$$(36) \quad f_v = \sum_{r=v}^n \frac{1}{P_r}, \quad F_v = \sum_{r=v}^n p_r f_r.$$

Let

$$(37) \quad v p_v = O(P_v),$$

$$(38) \quad P_v f_v = O(v).$$

Then sufficient conditions for the implication

$$(39) \quad \sum a_n \text{ is summable } |C, 1|_k \Rightarrow \sum a_n \lambda_n \text{ is summable } |(C, 1)(R, p_n)|_k$$

are

$$(40) \quad |\lambda_n| < n,$$

$$(41) \quad \nu p_\nu |\lambda_\nu| = O(P_\nu),$$

$$(42) \quad |\lambda_\nu| F_\nu = O(\nu),$$

$$(43) \quad |\Delta \lambda_\nu| F_\nu = O(1),$$

$$(44) \quad |\Delta \lambda_\nu| = O(1).$$

Proof: This follows from theorem 1, by putting $q_n = 1$ for all n, noticing that (7) for $q_n = 1$ is obviously satisfied as

$$\sum_{n=\nu+1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=\nu+1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{\nu}.$$

Corollary: 3.3 Let f_ν, F_ν be as defined in (27). Let (7) and (17) be satisfied and

$$(45) \quad \nu = O(Q_\nu).$$

Then necessary conditions for the implication (30) are

$$\lambda_\nu = O\left(\frac{Q_{\nu-1}}{1+F_\nu}\right), \lambda_\nu = O\left(\frac{\nu^{1-1/k} q_\nu}{f_\nu}\right) \quad \Delta \lambda_\nu = O\left(\frac{\nu^{1-1/k} q_\nu}{1+F_{\nu+1}}\right)$$

Proof: Follows from theorem 4 by putting $p_n = 1$ for all n.

Corollary: 3.4 Let f_ν, F_ν be as defined in (36). Let

$$(46) \quad P_\nu = O(\nu p_\nu).$$

Then necessary conditions for the implication (39) are

$$\lambda_\nu = O\left(\frac{\nu}{1+F_\nu}\right), \lambda_\nu = O\left(\frac{\nu^{1-1/k}}{P_\nu f_\nu}\right), \Delta \lambda_\nu = O\left(\frac{\nu^{1-1/k}}{1+F_{\nu+1}}\right).$$

Proof: Follows from theorem 4 by putting $q_n = 1$ for all n.

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