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A COMMON FIXED POINT THEOREM OF THREE SELFMAPS SATISFYING GENERALIZED GERAGHTY CONTRACTION CONDITION

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ABSTRACT

In this paper, we prove the existence of common fixed points for a generalized Geraghty contraction maps. Further we introduce generalized (f, g) - Geraghty contraction maps and prove the existence of common fixed points. Our results extend some of the known results. Examples are provided in support of our results.

keywords: *Common fixed points; Geraghty contraction; Generalized Geraghty contraction.*

AMS (2010) Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

Banach contraction principle is one of the fundamental results in fixed point theory for which several authors generalized and extended it by defining new contractive conditions. One among those conditions was due to Geraghty[6] namely Geraghty contraction, introduced in 1973, where the Geraghty contraction depends on the class of functions

$$S = \{\beta : [0, \infty) \to [0, 1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$$

Notation: We denote a metric space (X, d) by X. If $f : X \to X$ is a self map of X, we denote the set of all fixed points of f by F (f). i.e., $F(f) = \{x \in X : f(x) = x\}$.

If M is a nonempty sub set o f X then $cl[M]$ denotes the closure of M.

Definition 1.1: [6] Let (X, d) be a metric space. A selfmap $f : X \to X$ is said tobe a *Geraghty contraction* if there exists $\beta \in S$ such that

$$d(f(x), f(y)) \le \beta(d(x, y))d(x, y) \text{ for all } x, y \in X. \quad (1.1.1)$$

Here we observe that every contraction is a Geraghty contraction, but itsconverse need not be true [2], [3].

Theorem 1.2: [6] Let (X, d) be a complete metric space. Let $f : X \to X$ be aself map. If there exists $\beta \in S$ such that

$$d(f(x), f(y)) \le \beta(d(x, y))d(x, y) \text{ for all } x, y \in X, \quad (1.2.1)$$

then f has a unique common fixed point in X .

Definition 1.3: [3] A selfmap $f : X \to X$ is said to be a *generalized Geraghty contraction* if there exists $\beta \in S$ such that

$$d(f(x), f(y)) \le \beta(M(x, y))M(x, y) \quad (1.3.1)$$

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), (d(x, f(y)) + d(y, f(x))/2\} \text{ for all } x, y \in X.$$

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Suppose that M is a nonempty subset of a metric space (X, d) and f, T are self mappings of M . A point x in M to be *a common fixed* (*coincidence*) *point of* f and T if $x = fx = Tx$ ($fx = Tx$). We denote the set of coincidence points of f and T by $C(f, T)$, and the set of all common fixed points by $F(f, T)$.

i.e., $F(f, T) = F(f) \cap F(T)$.

Definition 1.4: [8] A pair (f, T) of selfmaps of a metric space (X, d) is said to be *weakly compatible*, if they commute at their coincidence points, *i.e.*, if $fTx = Tfx$ whenever $fx = Tx$ for all $x \in X$.

Definition 1.5: [9] Let K be a nonempty subset of a metric space (X, d) and T, f, g be three self maps on K. A mapping $T: K \to K$ is called generalized (f, g) - contraction if there exists a constant k [0, 1) such that

$d(Tx,Ty) \le k \max\{d(fx,gy), d(Tx,fx), d(TY,gy), \frac{d(fx,Ty)+ d(Tx,gy)}{2}\}$.

In 2007, Song [9] proved the following theorem for three maps. 1.6

Theorem 1.6: [9], [4] Let K be a metric space (x,d) and T, f, g : K →K three mappings with

Cl (T(K) C F(K) ∩ g(K) . *Suppose that Cl (T(K) is complete, T is generalized(f, g)- contraction with contestant* $k \in [0,1)$. If the pairs (T, f) and (T, g) are weakly compatible, then $F(T) \cap F(f) \cap F(g)$ is a singleton.

In 2013, Chandok and Narang [5] proved the following Theorem.

Theorem1.7: (Chandok [5]). Let M be a nonempty closed subset of a metric Space (X, d). Let T, f: M → M be self mappings q ∈F(f) and T(M\{q}) C f(M)\{q}. Suppose that there exists k ∈ [0, 1) such that

$$d(Tx,Ty) \le k \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}$$

For all x, y ∈M. Further, if T is continuous, cl[T(M\{q}] is complete, and f and T are weakly compatible on M\ {q} then F(f, T) is a singleton.

In section 2, we introduce generalized (f, g)- Geraghty contraction maps T and prove the existence of common fixed points for maps f, g and T by using technique of Chandok and Narang[5] by auuuming q ∈ $F(f) \cap F(g)$. Hence we note that q need not be same as the common fixed point of f , g and T.

Here we introduce (f-g) Geraghty contraction map T.

Definition 1.8: Let (X, d) be a metric space. A self map T: X → X is said to be generalized (f, g) Geraghty contraction if there exists $\beta \in S$ such that

$$d(Tx,Ty) \le \beta(M_1(x,y))M_1(x,y) \tag{1.8.1}$$

Where $M_1(x,y) = \max\{d(fx,gy), d(Tx,fx), d(Ty,gy), \frac{d(fx,TY)+d(gy,Tx)}{2}\}$ for all x, y ∈ X.

2. MAIN RESULTS

The following Lemma is useful in our subsequent discussions.

Lemma 2.1: Let M be a nonempty set and f: M →M be a selfmap. If q ∈F(f) then f(M) \{q} C f(M \{q}).

Proof: Let y ∈ f(M) \{q}. Then there exists x ∈ M, such that y=f(x), y≠q.

If x = q then f(x) = f(q) = q since q ∈ F(f).

i.e., y = f(x) = f(q) = q that implies y = q, a contradiction.

Therefore x ≠ q, so that x ∈ M\{q} and y=f(x) ∈ f(M\ {q}),y ≠ q.

Now, we prove the existence of common fixed point of generalized (f, g)-Geraghty type contraction.

Theorem 2.2: Let (X, d) be a metric space, M be a nonempty closed subset of X. Lef f, g and T be selfmaps of M. Let q ∈ $F(f) \cap F(g)$ and T(M\ {q}) C f(M \ {q} ∩ (g(M)\ {q}) and cl[T(M \ {q})] is complete. Suppose that there exists $\beta \in S$ such that

$$d(Tx,Ty) \le \beta(m_1(x,y))m_1(x,y) \tag{2.2.1}$$

Where $m_1(x,y) = \max\{d(fx,gy), d(Tx,fx), d(Ty,gy), \frac{d(fx,Ty)+d(gy,Tx)}{2}\}$ for all x, y ∈ M.

Further, if the pairs (T, f) and (T, g) are weakly compatible , then F(f) ∩ F(g) ∩ F(T) is singleton.

Proof: Let $x_0 \in M\backslash\{q\}$. Since T(M\ {q}) ⊂ f(M \ {q} ∩ (g(M) \ {q}), we can find $x_1,\ x_2 \in M\backslash\{q\}$ such that T $x_0 = f\ x_1$ and T $x_1 = g\ x_2$.

Similarly we can fond $x_3,\ x_4 \in M\backslash\{q\}$ such that T $x_2 = f\ x_3$ and T $x_3 = g\ x_4$.

On continuing this process, we can find $\{x_n\} \in M\backslash\{q\}$ such that T$x_{2n} = fx_{2n+1}$ and T$x_{2n+1} = gx_{2n+2}$ for n = 0, 1, 2, 3,….

Suppose that Tx_{2n} = T$x_{2n+1}$for some n .

Then we have $d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n+2}) \leq \beta\big(m_1(x_{2n+1}, x_{2n+2})\big)m_1(x_{2n+1}, x_{2n+2})$,
where

$$m_1(x_{2n+1}, x_{2n+2}) = \max\{d(fx_{2n+1}, gx_{2n+2}), d(Tx_{2n+1}, fx_{2n+1}), d(Tx_{2n+2}, gx_{2n+2}), \frac{d(fx_{2n+1},Tx_{2n+2})+d(gx_{2n+2},Tx_{2n+1})}{2}\}$$

$$= \max\{(Tx_{2n}, Tx_{2n+1}), (Tx_{2n}, Tx_{2n+1}), (Tx_{2n+1}, Tx_{2n+2}), \frac{d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})}{2}\}$$

$$= d(Tx_{2n+1}, Tx_{2n+2})$$

This implies that $d(Tx_{2n+1}, Tx_{2n+2}) < d(Tx_{2n+1}, Tx_{2n+2})$, a contradiction .

Hence $Tx_{2n+1} = Tx_{2n+2}$ so that $Tx_{2n} = Tx_{2n+1} = Tx_{2n+2}$.

Hence {Tx_{2n}} is a constant sequence, and hence it is a Cauchy.

Without loss of generality, we assume that $Tx_{2n} \neq Tx_{2n+1}$ for all n= 0, 1, 2,….

Now
We have $d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) \leq \beta\big(m_1(x_{2n+1}, x_{2n})\big)m_1(x_{2n+1}, x_{2n})$ (2.2.2)
where

$$m_1(x_{2n+1}, x_{2n}) = \max\{d(fx_{2n+1}, gx_{2n}), d(Tx_{2n+1}, fx_{2n+1}), d(Tx_{2n}, gx_{2n}), \frac{d(fx_{2n+1},Tx_{2n})+d(gx_{2n},Tx_{2n+1})}{2}\}$$

$$= \max\{d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Tx_{2n}), \frac{d(Tx_{2n},Tx_{2n})+d(Tx_{2n-1},Tx_{2n+1})}{2}\}$$

$$= \max\{ d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n}, Tx_{2n+1}), \frac{d(Tx_{2n-1},Tx_{2n+1})}{2}\}$$

$$= \max\{ d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n}, Tx_{2n+1})\}$$

Suppose that max { $d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n}, Tx_{2n+1})$}= (Tx_{2n+1}, Tx_{2n})

Then from (2.2.2), we have

$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) \leq \beta\big(m_1(x_{2n+1}, x_{2n})\big)m_1(x_{2n+1}, x_{2n})$, and hence \
$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) < d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n})$, *a contradiction.*

Hence max { $d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n}, Tx_{2n+1})$}= (Tx_{2n}, Tx_{2n-1})

Therefore from (2.2.2), we have

$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) \leq \beta\big(m_1(x_{2n+1}, x_{2n})\big)d(Tx_{2n}, Tx_{2n-1})$, and so
$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) < d(Tx_{2n}, Tx_{2n-1})$. (2.2.3)

Similarly it is easy to see that

$d(\mathrm{T}x_{2n-1}, \mathrm{T}x_{2n}) < d(Tx_{2n-2}, Tx_{2n-1})$. (2.2.4)

Hence from (2.2.3) and (2.2.4)

$d(\mathrm{T}x_{n+1}, \mathrm{T}x_n) < d(Tx_n, Tx_{n-1})$ for all n (2.2.5).

Thus $\{d(\mathrm{T}x_{n+1}, \mathrm{T}x_n)\}$ is a strctly decreasing sequence of non-negative real numbers and *so* $\lim_{n\to\infty} d(\mathrm{T}x_{n+1}, \mathrm{T}x_n)$ exists and is is r(say).

i.e., $\lim_{n\to\infty} d(\mathrm{T}x_{n+1}, \mathrm{T}x_n) = r, r \geq 0$.

Now from (2.2.2)n, we have

$d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) \leq \beta\big(m_1(x_{2n+1}, x_{2n})\big)m_1(x_{2n+1}, x_{2n})$

Suppose that $\beta\big(m_1(x_{2n+1}, x_{2n})\big) \to 1$ as $n \to \infty$, then by the hypothesis of β, $m_1(x_{2n+1}, x_{2n}) \to 0$ as $n \to \infty$. Then there exists $0 < \gamma < 1$ $such that$ $\beta\big(m_1(x_{2n+1}, x_{2n})\big) < \gamma$ $for\ infinetely\ many\ n.$
Then, $d(\mathrm{T}x_{2n+1}, \mathrm{T}x_{2n}) \le \gamma . m_1(x_{2n+1}, x_{2n})$

On letting $n \to \infty$, we have $r \le \gamma . r$.

Hence r = 0.

Hence $\lim_{n\to\infty} d(\mathrm{T}x_{n+1}, \mathrm{T}x_n) = 0$. (2.2.6)

Now, we show that $\{\mathrm{T}x_n\}$ $is\ Cauchy$.

By (2.2.5) and (2.2.6) , it is sufficient to show that $\{\mathrm{T}x_{2n}\}$ is Cauchy.

Suppose that $\{\mathrm{T}x_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and sequences of positive integers {n(k)} and {m(k)} with n(k) > m(k) > k, such that

$$d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) > \varepsilon \text{ and } d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-2}) \le \varepsilon. \quad (2.2.7)$$

$$\varepsilon \le \liminf \ d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) \quad (2.2.8)$$

By using Triangular inequality we have

$$d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) \le d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-2}) + d(\mathrm{T}x_{2n(k)-2}, \mathrm{T}x_{2n(k)}) + d(\mathrm{T}x_{2n(k)-1}, \mathrm{T}x_{2n(k)})$$

On taking limit supremum of both sides, as $k\to\infty$, we get

$$\lim_{k\to\infty} \sup d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) \le \varepsilon \quad (2.2.9)$$

Therefore from (2.2.8) and (2.2.9), we have

$$\lim_{k\to\infty} d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) \le \varepsilon \quad (2.2.10)$$

Now,

$$d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) \le d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-1}) + d(\mathrm{T}x_{2n(k)-1}, \mathrm{T}x_{2n(k)}) .$$

On taking limit infimum of both sides, as $k\to\infty$, we get

$$\varepsilon \le \lim_{k\to\infty} \inf d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-1}) \quad (2.2.11)$$

Again by using triangle inequality. we have

$$d\big(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-1}\big) \le d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)}) + d(\mathrm{T}x_{2n(k)}, \mathrm{T}x_{2n(k)-1})$$

On taking limit supremum of both sides, as $k\to\infty$, we get

$$\lim_{k\to\infty} \sup d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-1}) \le \varepsilon \quad (2.2.12)$$

Hence from (2.2.10) and (2.2.12), we have

$$\lim_{k\to\infty} d(\mathrm{T}x_{2m(k)}, \mathrm{T}x_{2n(k)-1}) = \varepsilon \quad (2.2.13)$$

Similarly we get

$$\lim_{k\to\infty} d(\mathrm{T}x_{2m(k)+1}, \mathrm{T}x_{2n(k)}) = \varepsilon \quad (2.2.14)$$

$$\lim_{k\to\infty} d(\mathrm{T}x_{2m(k)+1}, \mathrm{T}x_{2n(k)-1}) = \varepsilon \quad (2.2.15)$$

Now Consider

$$d(\mathrm{T}x_{2m(k)+1}, \mathrm{T}x_{2n(k)}) \le \beta\left(m_1\big(x_{2m(k)+1}, x_{2n(k)}\big)\right) m_1\big(x_{2m(k)+1}, x_{2n(k)}\big)$$

Where $m_1\big(x_{2m(k)+1}, x_{2n(k)}\big) = \max\{d\big(fx_{2m(k)+1}, gx_{2n(k)}\big), d\big(fx_{2m(k)+1}, Tx_{2m(k)+1}\big), d\left(gx_{2n(k)}, Tx_{2n}(k)\right),$

$$\frac{d\big(fx_{2m(k)+1}, Tx_{2n(k)}\big) + d\big(gx_{2m(k)}, Tx_{2m(k)+1}\big)}{2}\} \quad (2.2.16)$$

$$= \max\{d\big(Tx_{2m(k)}, Tx_{2n(k)-1}\big), d\big(Tx_{2m(k)+1}, Tx_{2m(k)+1}\big), , d\left(Tx_{2n(k)-1}, Tx_{2n}(k)\right),$$

$$\frac{d\big(Tx_{2m(k)+1}, Tx_{2n(k)}\big) + d\big(Tx_{2n(k)-1}, Tx_{2m(k)+1}\big)}{2}\}$$

On letting as k→ ∞, using (2.2.10), (2.2.13), (2.2.14) and (2.2.15)

$\lim_{k\to\infty} m_1(x_{2m(k)+1}, x_{2n(k)}) = \lim_{k\to\infty} = \max\{ d(Tx_{2m(k)}, Tx_{2n(k)-1}), d(Tx_{2m(k)+1}, Tx_{2m(k)+1}), d(Tx_{2n(k)-1}, Tx_{2n}(k)),$
$\frac{d(Tx_{2m(k)+1}, Tx_{2n(k)}) + d(Tx_{2n(k)-1}, Tx_{2m(k)+1})}{2}\}$ (2.2.17)

$= \max\{ \varepsilon, 0, 0, \frac{\varepsilon+\varepsilon}{2}\}$
$= \varepsilon$

From (2.2.16), we have

$$d(Tx_{2m(k)+1}, Tx_{2n(k)}) \le \beta\left(m_1(x_{2m(k)+1}, x_{2n(k)})\right) m_1(x_{2m(k)+1}, x_{2n(k)})$$

Taking limit as k→ ∞,

$$\lim_{k\to\infty} d(Tx_{2m(k)+1}, Tx_{2n(k)}) \le \lim_{k\to\infty} \beta\left(m_1(x_{2m(k)+1}, x_{2n(k)})\right) \lim_{k\to\infty} m_1(x_{2m(k)+1}, x_{2n(k)})$$
$$\varepsilon \le \lim_{k\to\infty} \beta\left(m_1(x_{2m(k)+1}, x_{2n(k)})\right)$$
$$\varepsilon \le \lim_{k\to\infty} \beta\left(m_1(x_{2m(k)+1}, x_{2n(k)})\right) \le 1$$

i.e., $\lim_{k\to\infty} \beta\left(m_1(x_{2m(k)+1}, x_{2n(k)})\right) = 1$, Since $\beta \in S$, *it follows that* $m_1(x_{2m(k)+1}, x_{2n(k)}) \to 0$, as k→ ∞.

So that$\varepsilon = 0$, a contradiction. Thus $\{Tx_{2n}\}$ is a Cauchy sequence and hence $\{Tx_n\}$ is a Cauchy sequence in M \ {q}.

Since cl [T(M\{q}] is complete , there exists u, v ∈ M \ {q} such that fu = z = gv.

Now we prove that Tu = z.

$$d(Tu, Tx_{2n}) \le \beta(m_1(u, x_{2n})) m_1(u, x_{2n})$$

where $m_1(u, x_{2n}) = \max\{d(fu, gx_{2n}) d(fu, Tu), d(Tx_{2n}, gx_{2n}), \frac{d(fu, Tx_{2n}) + d(gx_{2n}, Tu)}{2}\}$

$$\lim_{n\to\infty} m_1(u, x_{2n}) = \lim_{n\to\infty} \max\{d(fu, gx_{2n}) d(fu, Tu), d(Tx_{2n}, gx_{2n}), \frac{d(fu, Tx_{2n}) + d(gx_{2n}, Tu)}{2}\}$$
$$\lim_{n\to\infty} m_1(u, x_{2n}) = \max\{d(fu, z) d(fu, Tu), d(z, z), \frac{d(fu, z) + d(z, Tu)}{2}\}$$
$$= d(Tu, z).$$

Therefore $\lim_{n\to\infty} d(Tu, Tx_{2n}) \le \lim_{n\to\infty} \beta(m_1(u, x_{2n})) \lim_{n\to\infty} m_1(u, x_{2n})$

$$d(Tu, z) \le \lim_{n\to\infty} \beta(m_1(u, x_{2n})) d(Tu, z)$$
$$1 \le \lim_{n\to\infty} \beta(m_1(u, x_{2n})) \le 1$$

$i.e.,$ $\lim_{n\to\infty} \beta(m_1(u, x_{2n})) = 1$, Since β ∈ *S*, *by the hypothesis of* β,
$(m_1(u, x_{2n})) \to 0$, as n→ ∞ so that Tu=z.

$Therefore\ Tu = fu = z = gv = z.$

Since the pairs (T, f) and (T, g) are weakly compatible , we have Tu= fu implies Tfu = fTu = fz and hence gz= Tz=fz .

Therefore z is a common coincident point of f, g and T.

Now we claim that z is a common fixed point of f, g and T.

Suppose that $Tz \ne z$.

$$d(z, Tz) = d(Tu, Tz) \le \beta(m_1(u, z)) m_1(u, z) \quad (2.2.18)$$

Where $m_1(u, z) = \max\{ d(fu, gz), d(fu, Tu), d(Tz, gz), \frac{d(fu, Tz) + d(gz, Tu)}{2}\}$

$= \max \{ \max\{ d(z, gz), d(z, z), d(Tz, gz), \frac{d(z, Tz) + d(gz, Tu)}{2}\}$
$= d(z, Tz)$ (since gz= fz=Tz and Tu= fu=z)

Therefore from (2.2.18)

$$d(z, Tz) \le \beta(m_1(u, z)) d(z, Tz)$$
$$d(z, Tz) < d(z, Tz)$$, a contradiction.

$So\ that\ Tz = z. And\ hence\ Tz = fz = gz = z.$

$Uniqueness\ of\ z\ follows\ from\ the\ inequality$ (2.2.1),

$Therefore\ F(f)$ ∩ F(g) ∩ F(T) ≠ ϕ and $F(f)$ ∩ F(g) ∩ F(T) = {z}.

$\boldsymbol{Remark\ 2.3}$: $By\ choosing\ \ g = f\ in\ the\ inequality$ 2.2.1 $of\ Theorem$ 2.2 $, we\ the\ following\ corollary.$

Corollary 2.4: Let (X, d) be a metric space. M be a nonempty closed subset of X. Let f and T be selfmaps of M. Let q ∈ $F(f)$ and T(M\ {q}) ⊂ f(M \ {q} and cl[T(M \ {q})] is complete.

Suppose that there exists $\beta \in S$ such that

$$d(Tx, Ty) \le \beta(m(x, y))m(x, y) \tag{2.4 1}$$

Where $m(x, y) = \max\{d(fx, fy), d(Tx, fx), d(Ty, fy), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}$ for all x, y ∈ M.

Further, if the pairs (T, f) is weakly compatible, then F(f) ∩ F(T) is singleton.

Remark 2.5: Theorem 1.6 follows as a corollary to Theorem 2.2 by choosing

$\beta(t) = k,\ where\ k \in$[0,1) is of (1.5.1).

Remark 2.6: Theorem 1.7 follows as a corollary to Corollary 2.4 by choosing

$\beta(t) = k\ , where\ k \in$ [0, 1) is of (1.7.1).

Remark 2.7: In Theorem 2.2 the common fixed point q of f and g may not be common fixed point pf f, g and T. (Example 2.8).

The following is an example in support of Theorem 2.2 for simplicity, we take one of the maps f as the identity map in this example.

Example 2.8: Let X= M = [0, 1] with the usual metric. We define self maps f, g and T: M→ M by f(x) = x, g(x) = x^2 and $T(x) = \frac{x^2}{2}$.

We define $\beta : [0, \infty) \to [0, 1)$ $by\ \beta(t) = \frac{1}{1+t}$ for all t>0 and $\beta(0)=0$. $Then\ \beta \in$ S.

We choose q=1 ∈ F(f) ∩ F(g).

We now verify the inequality (2.2.1) for all x, y [0, 1].

$$d(Tx, Ty) = d\left(\frac{x^2}{2}, \frac{y^2}{2}\right) = |\frac{x^2-y^2}{2}|$$

$$m_1(x, y) = \max\{d(fx, gy), d(fx, Tx), d(gy, Ty), \frac{d(fx,Ty)+d(gy,Tx)}{2}\}$$

$$m_1(x, y) = \max\{d(x, y^2), d\left(x, \frac{x^2}{2}\right), d\left(y^2, \frac{y^2}{2}\right), \frac{d\left(x, \frac{y^2}{2}\right)+d\left(y^2, \frac{x^2}{2}\right)}{2}\}$$

$$m_1(x, y) = \max\{|x - y^2|, |x - \frac{x^2}{2}|, |y^2 - \frac{x^2}{2}|, \frac{|x-\frac{y^2}{2}|+|y^2-\frac{x^2}{2}|}{2}\}$$

$$|x - y^2| \le m_1(x, y).$$

$$d(Tx, Ty) = |\frac{x^2-y^2}{2}| \le \frac{(x-y^2)}{1+(x-y^2)} = \beta(m_1(x, y))m_1(x, y) .$$

$Theefore\ f, g\ and\ T\ satisfy\ all\ the\ hyptheses\ of\ Theorem$ 2.2 $and\ in\ fact$ '0' $is\ a\ commo\ fixed\ point\ of\ f\ , g\ and\ T.$

$i.e.,$ F(f) ∩ F(g) ∩ F(T) = 0 .

Here we observe that, 1 is a common foxed point of f and g, but it is not a common fixed point of f,g and T.(Remark 2.7)

The following is a supporting example of Corollary 2.4

Example 2.9: Let X= M= [0, 1] with the usual metric.

We define f: M→ M by f(x) = $\begin{cases} \frac{x}{2} & if\quad if\ 0 \le x \le \frac{1}{2} \\ x & if\quad \frac{1}{2} \le x \le 1. \end{cases}$

We define T: M→ M by T(x) = $\frac{x}{4}$.

We define β: [0, ∞) → [0, 1) by $\beta(t) = \frac{1}{1+t}$ for all t > 0 and $\beta(0)=0$. *Then* $\beta \in$ S.

We choose q = 1.

$$T(M\setminus \{q\}) = [0,\tfrac{1}{4}]\ C[0,\tfrac{1}{4}]\ U\ \left[\tfrac{1}{2},1\right] = f(M)\{q\}.$$

We now verify the inequality (2.4.1) in the following case.

Case-(i): Let x∈ [0, $\frac{1}{2}$], y ∈ [$\frac{1}{2}$, 1] .

In this case $d(Tx,Ty) = d\left(\frac{x}{4},\frac{y}{4}\right) = \frac{y-x}{4}$, and

$$M(x, y) = \max\left\{ d\left(\frac{x}{2},y\right), d\left(\frac{x}{4},\frac{x}{4}\right), d\left(\frac{x}{2}.\frac{y}{2}\right), \frac{d\left(\frac{x}{2},\frac{y}{2}\right)+d\left(y,\frac{x}{2}\right)}{2}\right\}$$
$$= \max\left\{ \frac{2y-x}{2}, \frac{x}{4}, \frac{y}{4}, \frac{6y-3x}{8}\right\}$$
$$= \frac{2y-x}{2}$$
$$d(Tx,Ty) = \frac{y-x}{4} \le \frac{1}{1+(\frac{2y-x}{2})}.\frac{2y-x}{2} = \beta\big(m(x,y)\big)m(x,y).$$

In the remaining case the inequality (2.4.1) holds trivially.

Corollary 2.4 and in fact it has a common fixed point ‘0’.

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