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ON THE ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

M. H. Gulzar*

Department of Mathematics, University of Kashmir, Srinagar-190006, India

E-mail: gulzarmh@gmail.com

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ABSTRACT

If P (z) be a polynomial of degree n with decreasing coefficients, then all its zeros lie in $|z|\le 1$. In this paper we present some generalizations of this result and a refinement of a result of Dewan and Bidkham.

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1. INTRODUCTION AND STATEMENT OF RESULTS:

The following elegant result in the theory of the distribution of the zeros of polynomials is due to Enestrom and Kakeya (see[9]-[10]).

Theorem: A Let P (z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$0< a_0 \le a_1 \le a_2 \le \ldots \le a_n,$$

then P(z) has all its zeros in $|z| \le 1$.

Several extensions and generalizations of this result are found in the literature (see[1]-[8],[11]) Aziz and Zargar ([2]) relaxed the hypothesis of Theorem A in an interesting way and proved the following extension of Theorem A:

Theorem: B Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some k ≥ 1,

$$ka_n \ge a_{n-1} \ge \ldots \ge a_1 \ge a_0 > 0,$$

then P(z) has all its zeros in $|z + k - 1| \le k$.

Dewan and Bidkham [4] proved the following result:

Theorem: C Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If Re $a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$,j=0,1,2,….,n such that

$$0 < \alpha_n \le \alpha_{n-1} \le \ldots \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \ldots \alpha_1 \ge \alpha_0 > 0,$$

where $0 \le \lambda \le n$, then all the zeros of P(z) lie in the circle

$$|z| \le \frac{1}{|a_n|}\{2\alpha_\lambda - \alpha_n + 2\sum_{j=0}^{n} |\beta_j|\}$$

Shah and Liman [11] also considered polynomials with complex coefficients and proved the following result:

**Corresponding author: M. H. Gulzar*,*E-mail: gulzarmh@gmail.com*

Theorem: D Let P(z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If Re $a_j = \alpha_j and$ $\operatorname{Im} a_j = \beta_j$,j=0,1,2,….,n such that for some k$\geq 1, 0 \leq \lambda \leq n-1$,

$$k\alpha_n \leq \alpha_{n-1} \leq ... \leq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ... \geq \alpha_1 \geq \alpha_{0,,}$$

$$\beta_n \geq \beta_{n-1} \geq ...\beta_{1\geq} \geq \beta_0 > 0,$$

then all the zeros of P(z) lie in

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{1}{|a_n|}\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}$$

Firstly, Theorem D is not correct, because k$\geq$1 has no meaning in the statement Moreover the authors claim that it is a generalization of Theorem B, which is absurd as the two results are in no way connected.

The purpose of this paper is to present the correct statement of Theorem D and then present an interesting generalization of the result which in particular provides a generalization and an extension of Theorem C .In Theorem D if we take k extremely large such that $k\alpha_n \geq \alpha_{n-1,}$ the hypothesis will not work .The correct statement of the theorem is as follows:

Theorem: 1.Let P(z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If Re $a_j = \alpha_j and$ $\operatorname{Im} a_j = \beta_j$,j=0,1,2,….,n such that for some $0 < k \leq 1, \quad ,0 \leq \lambda \leq n-1$,

$$k\alpha_n \leq \alpha_{n-1} \leq ... \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq ... \geq \alpha_1 \geq \alpha_{0,,}$$
$$\beta_n \geq \beta_{n-1} \geq ...\beta_{1\geq} \geq \beta_0 > 0,$$

then all the zeros of P(z) lie in

$$\left| z - \frac{\alpha_n}{a_n}(1-k) \right| \leq \frac{1}{|a_n|}\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}$$

As a generalization of this result, we shall prove the following result:

Theorem: 2 Let P (z)=$\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If Re$a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j, \ j = 0,1,2,..,n$ and for some real numbers $0 < k \leq 1, 0 < \tau \leq 1, \ 0 \leq \lambda \leq n-1,$

$$k\alpha_n \leq \alpha_{n-1} \leq ... \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq ... \geq \alpha_1 \geq \tau\alpha_{0,,}$$
$$\beta_n \geq \beta_{n-1} \geq ...\beta_{1\geq} \geq \beta_0 > 0,$$

then all the zeros of p (z) lie in

$$\left| z - \frac{\alpha_n}{a_n}(1-k) \right| \leq \frac{1}{|a_n|}\{2\alpha_\lambda - k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + \beta_n\}.$$

Corollary: 1 If the coefficients are >0 then, under the conditions of Theorem 2 , all the zeros of P(z) lie in

$$\left| z - \frac{\alpha_n}{a_n}(1-k) \right| \leq \frac{1}{a_n}\{2\alpha_\lambda - k\alpha_n + 2\alpha_0(1-\tau) + \beta_n\}$$

Taking k=1 in corollary 1, we get the following result:

Corollary: 2 If P(z) is a polynomial satisfying the conditions of Theorem 2, then all the zeros of P(z) lie in

$$|z| \le \frac{1}{a_n}\{2\alpha_\lambda + 2\alpha_0(1-\tau) + \beta_n - \alpha_n\}$$

Remark: 1 For k=1 and τ =1, corollary1 reduces to Theorem C due to Dewan and Bidkham.

Remark: 2 If all the coefficients of P(z) are real and τ =1 corollary 2 reduces to Enestrom- Kakeya Theorem.

Remark: 3.If the conditions of Theorem 2 are satisfied by the imaginary parts of the coefficients, then we are able to prove the following interesting result which follows by applying Theorem 2 to –iP(z).

Theorem: 3 Let Let P (z) = $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If.

$Re\ a_j = \alpha_j,\ Im\ a_j = \beta_j\ ,\ j = 0,1,2,..,n$ and for some real numbers

$0 < k \le 1\,, 0 < \tau \le 1,\ 0 \le \lambda \le n-1,$

$$\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0 \ge 0$$

$$k\beta_n \le \beta_{n-1} \le .... \le \beta_\lambda \ge \beta_{\lambda-1} \ge ...\beta_1 \ge \beta_0\tau\,,$$

then all the zeros of P(z) lie in

$$\left|z - \frac{\beta_n}{a_n}(1-k)\right| \le \frac{1}{|a_n|}\{2\beta_\lambda - k\beta_n + 2|\beta_0| - \tau(|\beta_0| + \beta_0) + \alpha_n\}\,.$$

2. PROOF OF THEOREM:

Proof of Theorem: 2 Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)\,(a_n z^n + a_{n-1}z^{n-1} + ...a_1 z + a_0)\\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ...(a_1 - a_0)z + a_0\\
&= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ...(\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + ...(\alpha_1 - \alpha_0)z + \alpha_0\\
&\quad + i\{(\beta_n - \beta_{n-1})z^n + ... + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}$$

$$\begin{aligned}
&= -a_n z^{n+1} - k\alpha_n z^n + \alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}\\
&\quad + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + ... + (\alpha_1 - \tau\alpha_0)z + (\tau-1)\alpha_0 z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + ... + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}$$

For |z|>1, we have

$$|F(z)| \ge \left[\begin{array}{l} |a_n z + k\alpha_n - \alpha_n| - \{|k\alpha_n - \alpha_{n-1}| + \dfrac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + ... + \dfrac{|\alpha_\lambda - \alpha_{\lambda+1}|}{|z|^{n-\lambda-1}} + \dfrac{|\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} \\ + ...\dfrac{|\alpha_1 - \tau\alpha_0|}{|z|^{n-1}} + .\dfrac{|1-\tau||\alpha_0|}{|z|^{n-1}} + .\dfrac{|\alpha_0|}{|z|^n}\} - \left\{|\beta_n - \beta_{n-1}| + ... + \dfrac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \dfrac{|\beta_0|}{|z|^n}\right\} \end{array}\right]$$

$$> |z|^n\left[|a_n z - (1-k)\alpha_n| - \left\{\begin{array}{l} -k\alpha_n + \alpha_{n+1} - \alpha_{n-1} + \alpha_{n-2} + ... + \alpha_\lambda - \alpha_{\lambda+1} + \alpha_\lambda - \alpha_{\lambda-1} + ... + \alpha_1 - \tau\alpha_0 \\ +(1-\tau)|\alpha_0| + |\alpha_0| + (\beta_n - \beta_{n-1}) + ...(\beta_1 - \beta_0) + \beta_0 \end{array}\right\}\right]$$

$$= |a_n||z|^n\left[\left|z - (1-k)\frac{\alpha_n}{a_n}\right| - \frac{1}{|a_n|}\{-k\alpha_n + 2\alpha_\lambda + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + \beta_n\}\right] > 0$$

If

$$\left|z-\frac{\alpha_n}{a_n}(1-k)\right|\leq\frac{1}{|a_n|}\{2\alpha_\lambda-k\alpha_n+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+\beta_n\}$$

This shows that the zeros of F (z) having modulus greater than 1 lie in

$$\left|z-\frac{\alpha_n}{a_n}(1-k)\right|\leq\frac{1}{|a_n|}\{2\alpha_\lambda-k\alpha_n+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+\beta_n\}$$

But those zeros of F (z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence, we conclude that all the zeros of F (z) lie in the disk

$$\left|z-\frac{\alpha_n}{a_n}(1-k)\right|\leq\frac{1}{|a_n|}\{2\alpha_\lambda-k\alpha_n+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+\beta_n\}$$

Since every zero of P (z) is also a zero of F (z) ,it follows that all the zeros of P(z)lie in the disk

$$\left|z-\frac{\alpha_n}{a_n}(1-k)\right|\leq\frac{1}{|a_n|}\{2\alpha_\lambda-k\alpha_n+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+\beta_n\}$$

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That completes the proof of the theorem 2.

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