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THE EQUITABLE BONDAGE NUMBER OF A GRAPH

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ABSTRACT

A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is called the equitable domination number and is denoted by $\gamma_e(G)$. We define the equitable bondage number $b_e(G)$ of a graph G to be the cardinality of a smallest set $X \subseteq E$ of edges for which $\gamma_e(G - X) > \gamma_e(G)$. Sharp bounds are obtained for $b_e(G)$ and the exact values are determined for some standard graphs.

Keywords: *Graph, Bondage number, Equitable bondage number, Equitable domination number, Equitable domination.*

Mathematics Subject Classification : *05C70.*

INTRODUCTION:

The graphs considered here are finite, undirected without loops and multiple edges having p vertices and q edges. Any undefined term in this paper may be found in Harary [3]. A set D of vertices in a graph G is a dominating set if each vertex of G that is not in D is adjacent to at least one vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For a survey of results on domination (see [4]). A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called the equitable domination number of G. The bondage number b(G) of G is the minimum cardinally among the sets of edges $X \subseteq E$ such that $\gamma(G - X) > \gamma(G)$ (see [2]). In this paper we now define the equitable bondage number of a graph G. The equitable bondage number $b_e(G)$ of a graph G is the minimum cardinality of a set $F \subseteq E$ of edges for which $\gamma_e(G - F) > \gamma_e(G)$.

SOME EXACT VALUES:

In several instances we shall have cause to use the ceiling function of a number x ; that is denoted by $\lceil x \rceil$ and takes the value of the least integer greater than or equal to x.

RESULTS:

Proposition 1: The equitable bondage number of the complete graph $K_p (p \geq 2)$ is $b_e(K_p) = \left\lceil \frac{p}{2} \right\rceil$.

Proof: If H is a spanning subgraph of K_p that is obtained by removing fewer than $\left\lceil \frac{p}{2} \right\rceil$ edges from K_p, then H contains a vertex of degree $p - 1$, whence $\gamma_e(H) = 1$. Thus $b_e(K_p) \geq \left\lceil \frac{p}{2} \right\rceil$.

We consider two cases.

Case 1: If p is even, the removal of $\frac{p}{2}$ independent edges from K_p reduces the degree of each vertex to $p - 2$ and therefore yields a graph H with equitable domination number $\gamma_e(H) = 2$.

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Case 2: If p is odd, the removal of $\frac{p-1}{2}$ independent edges from K_p leaves a graph having exactly one vertex of degree p – 1, by removing one edge incident with this vertex, we obtain a graph H with $\gamma_e(H) = 2$. In both cases, the graph H resulted from the removal of $\left\lceil \frac{p}{2} \right\rceil$ edges from K_p. Thus, $b_e(K_p) = \left\lceil \frac{p}{2} \right\rceil$.

We next determine the equitable bondage numbers of the cycle C_p and path P_p of order p.

Lemma A[1]: The equitable domination numbers of the cycle and path of order p are respectively

$$\gamma_e(C_p) = \left\lceil \frac{p}{3} \right\rceil \text{ for } p \geq 3 \text{ and}$$

$$\gamma_e(P_p) = \left\lceil \frac{p}{3} \right\rceil \text{ for } p \geq 1.$$

Theorem 2: Let $K_{m, n}$ be a complete bipartite graph with $|m – n| \leq 1$ and $m \leq n$ then $b_e(k_{m, n}) = m$.

Proof : Let $V = V_1 \cup V_2$ be the vertex set of $K_{m, n}$ where $|V_1| = m$ and $|V_2| = n$. Let $v \in V_2$ then by removing all edges incident with v we obtain a graph H containing two components K_1 and $K_{m, n-1}$. Hence

$$\begin{aligned} \gamma_e(H) &= \gamma_e(K_1) + \gamma_e(K_{m, n-1}) \\ &= 1 + \gamma_e(K_{m, n}) \\ &> \gamma_e(K_{m, n}) \end{aligned}$$

Thus,

$$\begin{aligned} b_e(K_{m, n}) &= \deg(v) \\ &= |V_1| \\ &= m. \end{aligned}$$

Theorem 3: The equitable bondage number of the p-cycle is

$$b_e(C_p) = \begin{cases} 3 & \text{if } p \equiv 1 \pmod 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof: Since $\gamma_e(C_p) = \gamma_e(P_p)$ for $p \geq 3$, we see that $b_e(C_p) \geq 2$. If $p \equiv 1 \pmod 3$ the removal of two edges from C_p leaves a graph H consisting of two paths X and Y. If X has order p_1 and Y has order p_2 then either $p_1 \equiv p_2 \equiv 2 \pmod 3$ or without loss of generality, $p_1 \equiv 0 \pmod 3$ and $p2 \equiv 1 \pmod 3$. In the former case,

$$\begin{aligned} \gamma_e(H) &= \gamma_e(X) + \gamma_e(Y) = \left\lceil \frac{p_1}{3} \right\rceil + \left\lceil \frac{p_2}{3} \right\rceil \\ &= \frac{(p_1+1)}{3} + \frac{(p_2+1)}{3} \\ &= \frac{p+2}{3} > \left\lceil \frac{p}{3} \right\rceil = \gamma_e(C_p) \end{aligned}$$

In the latter case,

$$\begin{aligned} \gamma_e(H) &= \frac{p_1}{3} + \frac{(p_2+2)}{3} \\ &= \frac{(p+2)}{3} > \left\lceil \frac{p}{3} \right\rceil = \gamma_e(C_p). \end{aligned}$$

In either case, when $p \equiv 1 \pmod 3$, we have $\gamma_e(C_p) \geq 3$. To obtain the upper bounds that by trichotomy, will yield the desired equalities of our theorem's statement we consider two cases.

Case 1: Suppose that $p \equiv 0, 2 \pmod 3$. The graph H obtained by removing two adjacent edges from C_p consists of an equitable isolated vertex and a path of order p – 1. Thus,

$$\gamma_e(H) = 1 + \gamma_e(P_{p-1}) = 1 + \left\lceil \frac{p-1}{3} \right\rceil$$

$$= 1 + \left\lceil \frac{p}{3} \right\rceil > \left\lceil \frac{p}{3} \right\rceil = \gamma_e(C_p)$$

hence $b_e(C_p) \leq 2$ in this case. Combining this with the upper bound obtained earlier, we have $b_e(C_p) = 2$ if $p \equiv 0, 2 \pmod 3$.

Case 2: Suppose now that $p \equiv 1 \pmod 3$. The graph H resulting from the deletion of three consecutive edges of C_p consists of two equitable isolated vertices and a path of order $n - 2$. Thus,

$$\gamma_e(H) = 2 + \left\lceil \frac{(p-2)}{3} \right\rceil = 2 + \frac{(p-1)}{3}$$
$$= 2 + \left\lceil \frac{p}{3} \right\rceil - 1 > \left\lceil \frac{p}{3} \right\rceil = \gamma_e(C_p)$$

so that $b_e(C_p) \leq 3$. With the earlier inequality, we conclude that $b_e(C_p) = 3$ when $p \equiv 1 \pmod 3$.

As an immediate corollary to the Theorem 3 we have the following

Corollary 3.1: The equitable bondage number of the path of order $(p \geq 2)$ is given by

$$b_e(P_p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 3 \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 4: The equitable bondage number of the wheel W_p is

$$b_e(W_p) = \begin{cases} 3 & \text{if } p \equiv 2 \pmod 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof: Let $W_p = K_1 + C_{p-1}$ and label C_{p-1}: $e_1, e_2 \ldots e_{p-1}$ be the edges of C_{p-1}. We consider two cases.

Case 1: Suppose that $p \equiv 2 \pmod 3$. The graph H obtained by removing three consecutive edges e_1, e_2, e_3 on C_{p-1} form W_p consists of three equitable isolated vertices and a path of order $p - 3$. Thus,

$$\gamma_e(H) = 3 + \gamma_e(P_{p-3})$$
$$= 3 + \left\lceil \frac{p-3}{3} \right\rceil > \gamma_e(W_p)$$

Hence, $b_e(W_p) = 3$.

Case 2: Suppose that $p \not\equiv 2 \pmod 3$. The graph H resulting from the deletion of two adjacent edges e_1 and e_2 on C_{p-1} from W_p consists of two equitable isolated vertices and a path of order $p - 2$. Thus,

$$\gamma_e(H) = 2 + \gamma_e(P_{p-2})$$
$$= 2 + \left\lceil \frac{p-2}{3} \right\rceil > \gamma_e(W_p)$$

Hence, $b_e(W_p) = 2$.

Theorem 5: If T is non trivial tree, then $b_e(T) \leq 2$.

Proof: If T is of order 2 or 3 then it is clear that $b_e(T) = 1$. We are interested here about the equitable edge in the tree. Suppose that T has at least four vertices, then we can classify trees into 2 cases.

Case 1: According to the end edge of the trees, if all the end edge are not equitable then by deleting any equitable edge the equitable domination number will be increasing i.e., $b_e(T) = 1$.

Case 2: If the end edge are equitable then we have two sub cases, if there exists leaf which is P_4 then $b_e(T) = 2$ otherwise $b_e(T) = 1$.

Hence for any nontrivial tree then $b_e(T) \leq 2$.

Theorem 6: If G be any graph and x be equitable edge then $\gamma(G - x) \geq \gamma(G)$.

Proof: Since we have a theorem, for any edge x of a graph, $\gamma(G - x) \geq \gamma(G)$ and from the definition of equitable domination number if the graph without any equitable isolated vertex then the $\gamma_e(G) = \gamma(G)$ and of course if we delete any equitable edge x from the graph G we get $\gamma(G - x) \geq \gamma(G)$.

Theorem 7: If G is a connected graph of order $p \geq 2$, then $b_e(G) \leq p - 1$.

Proof : Let u and v be adjacent vertices with $|deg(u) - deg(v)| \leq 1$ and $deg(u) \leq deg(v)$. Let E_u denote the set of edges incident with u. Then $\gamma_e(G - E_u) = \gamma_e(u)$ and $\gamma_e(G - u) = \gamma_e(G) - 1$. Also, if D denotes the union of all minimum equitable dominating sets for G – u, then u is adjacent in G to no vertex of D. Hence $|E_u| \leq p - 1 - |D|$ and $u \notin D$. Now if F_v denotes the set of edges from v to a vertex in D, then since $v \notin D$ we must have

$$\gamma_e(G - u - F_v) > \gamma_e(G - u) \text{ or equivalently,}$$
$$\gamma_e(G - u - F_v) > \gamma_e(G) - 1$$

Thus, $\gamma_e(G - (E_u \cup F_v)) > \gamma_e(G)$ and

we see that,
$$\begin{aligned} b_e(G) &\leq |E_u \cup F_v| \\ &= |E_u| + |F_v| \\ &\leq (p - 1 - |D|) + |D| \\ &= p - 1 \end{aligned}$$

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