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UNIQUE COMMON FIXED POINT THEOREMS FOR FOUR MAPS UNDER Ψ-Φ CONTRACTIVE CONDITION IN SYMMETRIC G – METRIC SPACES

K. P. R. Rao[1*], S. Hima Bindu[2] and J. Rajendra Prasad[3]

[1]Department of Applied Mathematics, Acharya Nagarjuna Univertsity-Dr. M. R. Appa Row Campus, Nuzvid- 521 201, Krishna District, Andhra Pradesh, India
E-mail: kprrao2004@yahoo.com

[2]Department of Mathematics, CH. S. D. St. Theresa's Junior College for women, Eluru- 534 001, W. G. Dt, A.P., India
E-mail: s.hima_bindu@yahoo.com

[3]Department of Information Technology, P. V. P. Siddhartha Institute of Technology, Kanur, Vijayawada-7, A.P., India
E-mail: rp.rajendra@rediffmail.com

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ABSTRACT

In this paper, we prove unique common fixed point theorems for four maps satisfying $\psi - \varphi$ contractive condition in symmetric G – metric spaces.

Keywords: *G – metric, symmetry, weakly compatible maps.*

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1. INTRODUCTION AND PRELIMINARIES:

Mustafa and Sims [16] and Naidu et.al [14] demonstrated that most of the claims concerning the fundamental topological structure of D – metric introduced by Dhage [1 – 5] and hence all theorems are incorrect. Alternatively, Mustafa and Sims [16, 17] introduced a G - metric space and obtained some fixed point theorems in it. Some interesting references in G - metric spaces are [8, 13, 15, 18 – 22].

Recently several authors are using the $\psi - \varphi$ contractive condition on maps to prove fixed and common fixed point theorems (See for example [9 – 12]).

In this paper, we prove two unique common fixed point theorems for four mappings satisfying $\psi - \varphi$ contractive condition in symmetric G – metric spaces. Before giving our main results, we recall some of the basic concepts and results in G – metric spaces.

Definition: 1.1 ([17]). Let X be a nonempty set and let $G: X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:
(G1): $G(x, y, z) = 0$ if $x = y = z$,
(G2): $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3): $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4): $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,
(G5): $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or a G – metric on X and the pair (X, G) is called a G–metric space.

Definition: 1.2 ([17]). The G – metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Corresponding author:** *K. P. R. Rao[1]*,E-mail:** kprrao2004@yahoo.com*

Definition: 1.3 ([17]). Let (X, G) be a G – metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be G – convergent to x.

Definition: 1.4 ([17]). Let (X, G) be a G – metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G – Cauchy iff $\lim_{l,n,m\to\infty} G(x_l, x_n, x_m) = 0$. (X, G) is called G – complete if every G – Cauchy sequence in (X, G) is G – convergent in (X, G).

Proposition: 1.5 ([17]). In a G – metric space, (X, G), the following are equivalent.
1. The sequence $\{x_n\}$ is G – Cauchy.
2. For every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition: 1.6 ([17]). Let (X, G) be a G – metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Proposition: 1.7 ([17]). Let (X, G) be a G – metric space. Then for any $x, y, z, a \in X$, it follows that
(i) if $G(x, y, z) = 0$ then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(x, x, y)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}\,[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

Proposition: 1.8 ([17]). Let (X, G) be a G – metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent
(i) $\{x_n\}$ is G – convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition: 1.9 ([6]). A pair of self mappings is called weakly compatible if they commute at their coincidence points.

Now we give our main results.

2. MAIN RESULTS:

Let Ψ denote the set of all continuous mappings $\psi: [0, \infty) \to [0, \infty)$.

Let Φ denote the set of all lower semi continuous mappings $\varphi: [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for all $t > 0$.

Theorem 2.1 . Let (X, G) be a symmetric G – metric space and S, T, f, g : X → X be satisfying

(2.1.1) $$\psi(G(Sx, Ty, z)) \leq \psi\left(\max\left\{\begin{matrix} G(fx, gy, z), G(fx, Sx, z), G(gy, Ty, z), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right) - \phi\left(\max\left\{\begin{matrix} G(fx, gy, z), G(fx, Sx, z), G(gy, Ty, z), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right)$$

$\forall x, y, z \in X$ with $z = Sx$ or Ty, where $\psi \in \Psi$, $\varphi \in \Phi$,

(2.1.2) $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$,

(2.1.3) (S, f) and (T, g) are weakly compatible,

(2.1.4) one of f(X) and g(X) is G – complete.

Then f, g, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Sx_{2n} = gx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = fx_{2n+2}$, n = 0, 1, 2...

Suppose $y_{2m+1} = y_{2m}$ for some m. Let $y_{2m+1} \neq y_{2m+2}$.

$\psi\,(G(y_{2m+2}, y_{2m+1}, y_{2m+1})) = \psi\,(G(Sx_{2m+2}, Tx_{2m+1}, y_{2m+1}))$

$$\leq \psi\left(\max\left\{\begin{matrix} G(y_{2m+1}, y_{2m}, y_{2m+1}), G(y_{2m+1}, y_{2m+2}, y_{2m+1}), G(y_{2m}, y_{2m+1}, y_{2m+1}), \\ G(y_{2m+1}, y_{2m+2}, y_{2m+2}), G(y_{2m}, y_{2m+1}, y_{2m+1}) \end{matrix}\right\}\right)$$

$$-\varphi\left(\max\left\{\begin{matrix} G(y_{2m+1}, y_{2m}, y_{2m+1}), G(y_{2m+1}, y_{2m+2}, y_{2m+1}), G(y_{2m}, y_{2m+1}, y_{2m+1}), \\ G(y_{2m+1}, y_{2m+2}, y_{2m+2}), G(y_{2m}, y_{2m+1}, y_{2m+1}) \end{matrix}\right\}\right)$$

$= \psi\,(G(y_{2m+1}, y_{2m+2}, y_{2m+1})) - \varphi\,(G(y_{2m+1}, y_{2m+2}, y_{2m+1}))$, since X is symmetric

$< \psi\,(G(y_{2m+1}, y_{2m+2}, y_{2m+1}))$.

It is a contradiction. Hence $y_{2m+1} = y_{2m+2}$.

Continuing in this way, we get $y_n = y_{n+k}$ for all k > 0. Hence $\{y_n\}$ is a G – Cauchy sequence in X.

Now assume that $y_n \neq y_{n+1}$ for all n.

Denote $p_n = G(y_n, y_{n+1}, y_{n+1})$.

$\psi\,(p_{2n}) = \psi\,(G(y_{2n}, y_{2n+1}, y_{2n+1}))$

$= \psi\,(G(y_{2n}, y_{2n+1}, y_{2n}))$, since X is symmetric

$= \psi\,(G(Sx_{2n}, Tx_{2n+1}, y_{2n}))$

$$\leq \psi\left(\max\left\{\begin{matrix} G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n}), \\ G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{matrix}\right\}\right)$$

$$-\varphi\left(\max\left\{\begin{matrix} G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n}), \\ G(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{matrix}\right\}\right)$$

$= \psi\,(\max\{p_{2n-1}, p_{2n}\}) - \varphi\,(\max\{p_{2n-1}, p_{2n}\})$, since X is symmetric.

If $\max\{p_{2n-1}, p_{2n}\} = p_{2n}$, then

$\psi\,(p_{2n}) \leq \psi\,(p_{2n}) - \varphi(p_{2n}) < \psi\,(p_{2n})$. It is a contradiction.

Hence $p_{2n} \leq p_{2n-1}$ and $\psi(p_{2n}) \leq \psi\,(p_{2n-1}) - \varphi(p_{2n-1})$ (I)

Similarly, we have $p_{2n+1} \leq p_{2n}$.

Thus $\{p_n\}$ is a decreasing sequence of non – negative real numbers and hence converge to some real number, $l \geq 0$.

Letting $n \to \infty$ in (I), we get

$\psi\,(l) \leq \psi(l) - \varphi(l)$ so that $\varphi(l) \leq 0$.

Hence $l = 0$. Thus

$$\lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+1}) = 0 \qquad \text{(II)}$$

Now we prove that $\{y_{2n}\}$ is a G - Cauchy sequence. Suppose it is not true. Then there exists an $\varepsilon > 0$ such that for any positive integer k, there exist positive integers $m_k > n_k > k$ such that

$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \geq \varepsilon$ (III)

and $G(y_{2n_k}, y_{2m_k-2}, y_{2m_k-2}) < \varepsilon$ (IV)

From (III),

$\varepsilon \leq G(y_{2n_k}, y_{2m_k}, y_{2m_k})$

$\leq G(y_{2m_k-2}, y_{2m_k}, y_{2m_k}) + G(y_{2n_k}, y_{2m_k-2}, y_{2m_k-2})$

$< G(y_{2m_k-2}, y_{2m_k}, y_{2m_k}) + \varepsilon$ from (IV)

$= G(y_{2m_k}, y_{2m_k-2}, y_{2m_k-2}) + \varepsilon$, since X is symmetric

$\leq G(y_{2m_k-1}, y_{2m_k-2}, y_{2m_k-2}) + G(y_{2m_k}, y_{2m_k-1}, y_{2m_k-1}) + \varepsilon$

$= G(y_{2m_k-2}, y_{2m_k-1}, y_{2m_k-1}) + G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}) + \varepsilon$, since X is symmetric

$= p_{2m_k-2} + p_{2m_k-1} + \varepsilon$.

Letting $k \to \infty$ and using (II), we get

$\lim_{k\to\infty} G(y_{2n_k}, y_{2m_k}, y_{2m_k}) = \varepsilon$ (V)

$G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) \leq G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) + G(y_{2m_k}, y_{2n_k}, y_{2n_k})$

$G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) - G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \leq p_{2n_k}$.

Also

$G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \leq G(y_{2n_k+1}, y_{2n_k}, y_{2n_k}) + G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1})$.

$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) - G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) \leq p_{2n_k}$.

Thus $|G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) - G(y_{2n_k}, y_{2m_k}, y_{2m_k})| \leq p_{2n_k}$.

Letting $k \to \infty$ and using (II) and (V), we get

$\lim_{k\to\infty} G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) = \varepsilon$ (VI)

$G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) \leq G(y_{2n_k}, y_{2n_k}, y_{2n_k+1}) + G(y_{2m_k-1}, y_{2n_k}, y_{2n_k})$

$\leq G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) + G(y_{2m_k}, y_{2n_k}, y_{2n_k}) + G(y_{2m_k-1}, y_{2m_k}, y_{2m_k})$, since X is symmetric.

$G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) - G(y_{2n_k}, y_{2m_k}, y_{2m_k}) \leq p_{2n_k} + p_{2m_k-1}$, since X is symmetric.

Also

$G(y_{2m_k}, y_{2n_k}, y_{2n_k}) \leq G(y_{2m_k}, y_{2n_k}, y_{2n_k+1}) + G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1})$

$\leq G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) + G(y_{2m_k}, y_{2m_k-1}, y_{2m_k-1}) + p_{2n_k}$.

$G(y_{2n_k}, y_{2m_k}, y_{2m_k}) - G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) \leq p_{2m_k-1} + p_{2n_k}$, since X is symmetric.

Thus $|G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) - G(y_{2n_k}, y_{2m_k}, y_{2m_k})| \leq p_{2m_k-1} + p_{2n_k}$.

Letting $k \to \infty$ and using (II) and (V), we get

$\lim_{k\to\infty} G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}) = \varepsilon$ (VII)

$G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}) \le G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}) + G(y_{2n_k+1}, y_{2m_k}, y_{2m_k})$.

$G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}) - G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) \le p_{2m_k-1}$, since X is symmetric.

Also

$G(y_{2n_k+1}, y_{2m_k}, y_{2m_k}) \le G(y_{2n_k+1}, y_{2m_k}, y_{2m_k-1}) + G(y_{2m_k}, y_{2m_k-1}, y_{2m_k-1})$.

$G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1}) - G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}) \le p_{2m_k-1}$, since X is symmetric.

Thus $|G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}) - G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1})| \le p_{2m_k-1}$.

Letting $k \to \infty$ and using (II) and (VI), we get

$$\lim_{k\to\infty} G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}) = \varepsilon \qquad \text{(VIII)}$$

Now

$\psi(G(y_{2m_k}, y_{2n_k+1}, y_{2n_k+1})) = \psi(G(Sx_{2m_k}, Tx_{2n_k+1}, y_{2n_k+1}))$

$$\le \psi\left(\max\left\{\begin{matrix} G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}), G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}), G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) \\ G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}), G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) \end{matrix}\right\}\right)$$
$$-\varphi\left(\max\left\{\begin{matrix} G(y_{2m_k-1}, y_{2n_k}, y_{2n_k+1}), G(y_{2m_k-1}, y_{2m_k}, y_{2n_k+1}), G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) \\ G(y_{2m_k-1}, y_{2m_k}, y_{2m_k}), G(y_{2n_k}, y_{2n_k+1}, y_{2n_k+1}) \end{matrix}\right\}\right)$$

Letting $k \to \infty$ and using (VI), (VII),(VIII) and (II), we get

$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$.

It is a contradiction. Hence $\{y_{2n}\}$ is G – Cauchy.

Hence $\lim_{n,m\to\infty} G(y_{2n}, y_{2m}, y_{2m}) = 0$.

Now

$0 \le G(y_{2n+1}, y_{2m+1}, y_{2m+1})$

$\le G(y_{2n}, y_{2m+1}, y_{2m+1}) + G(y_{2n+1}, y_{2n}, y_{2n})$

$\le G(y_{2n}, y_{2m+1}, y_{2m}) + G(y_{2m+1}, y_{2m}, y_{2m}) + p_{2n}$, since X is symmetric

$\le G(y_{2n}, y_{2m}, y_{2m}) + G(y_{2m+1}, y_{2m}, y_{2m}) + p_{2m} + p_{2n}$, since X is symmetric

$\le G(y_{2n}, y_{2m}, y_{2m}) + p_{2m} + p_{2m} + p_{2n}$, since X is symmetric.

Letting $n, m \to \infty$, we get

$0 \le \lim_{n,m\to\infty} G(y_{2n+1}, y_{2m+1}, y_{2m+1}) \le 0$.

Thus $\{y_{2n+1}\}$ is G – Cauchy. Hence $\{y_n\}$ is G – Cauchy.

Suppose f(X) is G – complete.

Then there exist $p, t \in X$ such that $y_{2n+1} \to p = fu$.

Since $\{y_n\}$ is G – Cauchy, it follows that $\{y_{2n}\} \to p$ as $n \to \infty$.

Now

$$\psi(G(Su, Tx_{2n+1}, Su)) \le \psi\left(\max\left\{\begin{matrix} G(fu, y_{2n}, Su), G(fu, Su, Su), G(y_{2n}, y_{2n+1}, Su), \\ G(fu, Su, Su), G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fu, y_{2n}, Su), G(fu, Su, Su), G(y_{2n}, y_{2n+1}, Su), \\ G(fu, Su, Su), G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{matrix}\right\}\right).$$

Letting $n \to \infty$, we get

$$\psi(G(Su, p, Su)) \le \psi(\max\{G(p, p, Su), G(p, Su, Su), G(p, p, Su), G(p, Su, Su), 0\}) - \varphi(\max\{G(p, p, Su), G(p, Su, Su), G(p, p, Su), G(p, Su, Su), 0\})$$

$\psi(G(Su, p, Su)) = \psi(G(Su, p, Su)) - \varphi(G(Su, p, Su))$, since X is symmetric

$< \psi(G(Su, p, Su))$ if $Su \ne p$.

Hence $Su = p$.

Thus $fu = p = Su$.

Since $S(X) \subseteq g(X)$, there exists $v \in X$ such that $Su = gv$.

$$\psi(G(Su, Tv, Su)) \le \psi(\max\{0, 0, G(Su, Tv, Su), 0, G(Su, Tv, Tv)\}) - \varphi(\max\{0, 0, G(Su, Tv, Su), 0, G(Su, Tv, Tv)\})$$

$= \psi(G(Su, Tv, Su)) - \varphi(G(Su, Tv, Su))$, since X is symmetric

$< \psi(G(Su, Tv, Su))$ if $Su \ne Tv$.

Hence $Su = Tv$.

Thus $fu = Su = Tv = gv = p$.

Since the pairs (S, f) and (T, g) are weakly compatible, we have

$fp = Sp$ and $gp = Tp$.

$$\psi(G(Sp, Tv, Su)) \le \psi\left(\max\left\{\begin{matrix} G(fp, gv, Su), G(fp, Sp, Su), G(gv, Tv, Su), \\ G(fp, Sp, Sp), G(gv, Tv, Tv) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fp, gv, Su), G(fp, Sp, Su), G(gv, Tv, Su), \\ G(fp, Sp, Sp), G(gv, Tv, Tv) \end{matrix}\right\}\right)$$

$$\psi(G(Sp, p, p)) \le \psi(\max\{G(Sp, p, p), G(Sp, Sp, p), 0, 0, 0\}) - \varphi(\max\{G(Sp, p, p), G(Sp, Sp, p), 0, 0, 0\})$$

$= \psi(G(Sp, p, p)) - \varphi(G(Sp, p, p))$, since X is symmetric

$< \psi(G(Sp, p, p))$ if $Sp \ne p$.

Hence $Sp = p$. Thus $fp = Sp = p$.

$$\psi(G(Su, Tp, Tv)) \le \psi\left(\max\left\{\begin{matrix} G(fu, gp, Tv), G(fu, Su, Tv), G(gp, Tp, Tv), \\ G(fu, Su, Su), G(gp, Tp, Tp) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fu, gp, Tv), G(fu, Su, Tv), G(gp, Tp, Tv), \\ G(fu, Su, Su), G(gp, Tp, Tp) \end{matrix}\right\}\right)$$

$$\psi(G(p, Tp, p)) \le \psi(\max\{G(p, Tp, p), 0, G(Tp, Tp, p), 0, 0\}) - \psi(\max\{G(p, Tp, p), 0, G(Tp, Tp, p), 0, 0\})$$

$= \psi(G(p, Tp, p)) - \varphi(G(p, Tp, p))$, since X is symmetric

$< \psi(G(p, Tp, p))$ if $Tp \ne p$.

Hence Tp = p.

Thus p is a common fixed point of f, g, S and T.

Suppose p^1 is another common fixed point of f, g, S and T.

$$\psi(G(p, p^1, p)) = \psi(G(Sp, Tp^1, Sp))$$

$$\le \psi(\max\{G(p, p^1, p), 0, G(p^1, p^1, p), 0, 0\}) - \varphi(\max\{G(p, p^1, p), 0, G(p^1, p^1, p), 0, 0\})$$

$$= \psi(G(p, p1, p)) - \varphi(G(p, p1, p)), \text{ since X is symmetric}$$

$$< \psi(G(p, p^1, p)) \text{ if } p \ne p^1.$$

Hence $p = p^1$.

Thus p is the unique common fixed point of f, g, S and T.

Similarly the theorem holds whenever g(X) is complete.

Definition: 2.2 ([7]) The pair (S, f) is said to be occasionally weakly compatible (owc) if there exists x ∈ X such that fx = Sx and fSx = Sfx.

Theorem: 2.3 Let (X, G) be a symmetric G – metric space and S, T, f, g : X → X be satisfying

(2.3.1) $$\psi(G(Sx, Ty, z)) \le \psi\left(\max\left\{\begin{matrix} G(fx, gy, z), G(fx, Sx, z), G(gy, Ty, z), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fx, gy, z), G(fx, Sx, z), G(gy, Ty, z), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right)$$

∀ x, y, z ∈ X with z = Sx or Ty, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ and $\varphi(t) > 0$ ∀ t > 0.

Also assume that (2.3.2) the pairs (S, f) and (T, g) are owc.

Then S, T, f and g have a unique common fixed point in X.

Proof: From (2.3.2), there exist u, v ∈ X such that fu = Su, Sfu = fSu and gv = Tv, Tgv = gTv.

Suppose Su ≠ Tv. Then

$$\psi(G(Su, Tv, Tv)) \le \psi\left(\max\left\{\begin{matrix} G(fu, gv, Tv), G(fu, Su, Tv), G(gv, Tv, Tv), \\ G(fu, Su, Su), G(gv, Tv, Tv) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fu, gv, Tv), G(fu, Su, Tv), G(gv, Tv, Tv), \\ G(fu, Su, Su), G(gv, Tv, Tv) \end{matrix}\right\}\right)$$

$$= \psi(G(Su, Tv, Tv)) - \varphi(G(Su, Tv, Tv)), \text{ since X is symmetric}$$

$$< \psi(G(Su, Tv, Tv)).$$

It is a contradiction. Hence Su = Tv.

Thus fu = Su = Tv = gv. Suppose $S^2u \ne Su$.

Then

$$\psi(G(S^2u, Tv, Su)) \le \psi\left(\max\left\{\begin{matrix} G(fSu, gv, Su), G(fSu, S^2u, Su), G(gv, Tv, Su), \\ G(fSu, S^2u, S^2u), G(gv, Tv, Tv) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fSu, gv, Su), G(fSu, S^2u, Su), G(gv, Tv, Su), \\ G(fSu, S^2u, S^2u), G(gv, Tv, Tv) \end{matrix}\right\}\right)$$

$= \psi(G(S^2u, Tv, Su)) - \varphi(G(S^2u, Tv, Su))$, since X is symmetric

$< \psi(G(S^2u, Tv, Su))$.

It is a contradiction. Hence $S^2u = Su$.

Thus $S(Su) = Su$......(i) and $f(Su) = Sfu = S^2u = Su$....(ii).

Suppose $T^2v \neq Tv$. Then

$$\psi(G(Su, T^2v, Tv)) \le \psi\left(\max\left\{\begin{matrix} G(fu, gTv, Tv), G(fu, Su, Tv), G(gTv, T^2v, Tv), \\ G(fu, Su, Su), G(gTv, T^2v, T^2v) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fu, gTv, Tv), G(fu, Su, Tv), G(gTv, T^2v, Tv), \\ G(fu, Su, Su), G(gTv, T^2v, T^2v) \end{matrix}\right\}\right)$$

$= \psi(G(Su, T^2v, Tv)) - \varphi(G(Su, T^2v, Tv))$, since X is symmetric

$< \psi(G(Su, T^2v, Tv))$.

It is a contradiction. Hence $T^2v = Tv$.

Thus $T(Su) = T(Tv) = Tv = Su$......(iii) and $g(Su) = gTv = Tgv = T^2v = Tv = Su$......(iv).

From (i), (ii), (iii) and (iv), it follows that Su is a common fixed point of S, T, f and g.

Uniqueness of common fixed point follows easily from (2.3.1).

The following examples illustrate the Theorems 2.1 and 2.3.

Example: 2.4. Let $X = [0, 1]$ and $G: X \times X \times X \to [0, \infty)$ be defined by

$G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$.

Let $S, T, f, g: X \to X$ be defined by $Sx = 0$, $Tx = \frac{x}{8}$, $fx = x$, $gx = \frac{x}{2}$, for all $x \in X$.

Let $\psi, \varphi: [0, \infty) \to [0, \infty)$ be defined by $\psi(t) = t$, $\varphi(t) = \frac{3t}{4}$, for all $t \ge 0$.

Now, $G(Sx, Ty, Sx) = G(0, \frac{y}{8}, 0) = \frac{y}{4}$

and $G(gy, Ty, Sx) = G(\frac{y}{2}, \frac{y}{8}, 0) = y$.

$$\begin{aligned} \psi(G(Sx, Ty, Sx)) &= G(Sx, Ty, Sx) \\ &= \frac{y}{4} \\ &= \frac{1}{4}\psi\, G(gy, Ty, Sx) \\ &\le \frac{1}{4}\max\{G(fx, gy, Sx), G(fx, Sx, Sx), G(gy, Ty, Sx), G(fx, Sx, Sx), G(gy, Ty, Ty)\} \\ &= \psi\left(\max\left\{\begin{matrix} G(fx, gy, Sx), G(fx, Sx, Sx), G(gy, Ty, Sx), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right) - \varphi\left(\max\left\{\begin{matrix} G(fx, gy, Sx), G(fx, Sx, Sx), G(gy, Ty, Sx), \\ G(fx, Sx, Sx), G(gy, Ty, Ty) \end{matrix}\right\}\right). \end{aligned}$$

Thus (2.1.1) or (2.3.1) is satisfied with z = Sx.

One can easily verify the remaining conditions in Theorem 2.1 and Theorem 2.3. Clearly 0 is the unique common fixed point of S, T, f and g.

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