On the Structure of Some Groups Containing $L_2(13) \, wr \, L_2(17)$

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ABSTRACT

In this paper, we will generate the wreath product $L_2(13) \, wr \, L_2(17)$ using only two permutations. Also, we will show the structure of some groups containing the wreath product $L_2(13) \, wr \, L_2(17)$. The structure of the groups founded is determined in terms of wreath product $(L_2(13) \, wr \, L_2(17)) \, wr \, C_k$. Some related cases are also included. Also, we will show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $(L_2(13) \, wr \, L_2(17)) \, wr \, C_k$ and a transposition in S_{252k+1} and an element of order 3 in A_{252K+1}. We will also show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $L_2(13) \, wr \, L_2(17)$ and an element of order $k+1$.

Keywords and phrases: wreath product, Linear group.
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1. INTRODUCTION:

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1}. They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5].

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product A_m wr S_a and an element of order $k+1$. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using n elements each of order $k+1$.

Al-Amri and Al-Shehri [3] have shown that S_{9k+1} and A_{9k+1} can be generated using the wreath product $M_9 wr C_k$ and an element of order 4 in S_{9k+1} and element of order 5 in A_{9k+1}.

The Linear groups $L_2(13)$ and $L_2(17)$ are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows:

$$L_2(13) = < X,Y \mid X^{13} = Y^2 = (X^4 Y X^7 Y)^2 = (XY)^3 = 1 >$$
$$L_2(17) = < X,Y \mid X^{17} = Y^2 = (X^4 Y X^9 Y)^2 = (XY)^3 = 1 >$$

$L_2(13)$ can be generated using two permutations, the first is of order 13 and an involution as follows:
$L_2(13) = < (1,2,..,13)(1,5)(3,4)(6,8)(7,14)(9,13)(10,11) >$.

$L_2(17)$ can be generated using two permutations, the first is of order 17 and an involution as follow: $L_2(17) = < (1,2,...,17)(1,16)(2,8)(3,11)(5,10)(6,14)(7,12)(9,15)(17,18) >$

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In this paper, we will generate the wreath product $L_2(13) \, wr \, L_2(17)$ using only two permutations. Also, we show the structure of some groups containing the wreath product $L_2(13) \, wr \, L_2(17)$. The structure of the groups founded is

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determined in terms of wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$. Some related cases are also included. Also, we will show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and a transposition in S_{252k+1} and an element of order 3 in A_{252K+1}. We will also show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $L_2(13)\,wr\,L_2(17)$ and an element of order $k+1$.

2. PRELIMINARY RESULTS:

Definition: 2.1 Let A and B be groups of permutations on non empty sets Ω_1 and Ω_2 respectively. The wreath product of A and B is denote by A wr B and defined as A wr $B = A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of A and a mapping θ where $\theta : B \to \mathrm{Aut}\,(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that $|A$ wr $B| = (|A|)^{|\Omega_2|}|B|$.

Theorem: 2.2 [4] Let G be the group generated by the n-cycle $(1, 2\ldots n)$ and the 2-cycle (n, a). If $1 < a < n$ is an integer with $n = am$, then $G \cong S_m$ wr C_a.

Theorem 2.3 [4] Let $1 \le a \ne b < n$ be any integers. Let n be an odd integer and let G be the group generated by the n-cycle $(1,2,\ldots,n)$ and the 3-cycle (n,a,b). If the $hcf(n,a,b)=1$, then $G = A_n$. While if n can be an even then $G = S_n$.

Theorem: 2.4 [4] Let $1 \le a < n$ be any integer. Let $G = \langle\,(1,2,\ldots,n)\,,\,(n,a)\,\rangle$. If $h.c.f.\,(n,a) = 1$, then $G = S_n$.

Theorem: 2.5 [4] Let $1 \le a \ne b < n$ be any integers. Let n be an even integer and let G be the group generated by the $(n\text{-}1)$-cycle $(1,2,\ldots,n-1)$ and 3-cycle (n,a,b). Then $G = A_n$.

3. THE RESULTS:

Theorem: 3.1 The wreath product $L_2(13)\,wr\,L_2(17)$ can be generated using two permutations, the first is of order 252 and the second is of order 4.

Proof: Let $G = \langle X,Y \rangle$, where: X=(1, 2, 3, 4, …, 252), which is a cycle of order 252, Y=(1, 9)(2, 6)(4, 5)(7, 8) (12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27) (29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53) (46, 50)(48, 49) (51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = \left((XY)^6[X,Y]^5\right)^{18}$. Then

$$\alpha_1 = (17, 22, 33, 44, 55, 66, 252),$$

which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1}X$. It is easy to show that

$$\alpha_2 = (1, 2, 3\ldots 17)\ (18, 19, 20\ldots 22)\ \ldots\ (67, 68, 69\ldots 252),$$

which is the product of seven cycles each of order 17. Let: $\beta_1 = (Y^2)^{(XY)^{18}}$ =(9, 20)(12, 23)(31, 53)(34, 56),

$\beta_2 = \beta_1 Y^{-1}$ =(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74),

$\beta_3 = (Y^3\beta_2)^2$ =(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(11, 44)(55, 66) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (17, 221)(68, 85). Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$.

Hence

$$\alpha_3 = (17, 34)\ (51, 85).$$

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

$\alpha_4 = (1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)$
$(46,50)(48,49)(51,52)(56,64)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74),$

which is the product of twenty eight transpositions. Let $K = \langle \alpha_2, \alpha_4 \rangle$. Let $\theta : K \to L_2(17)$ be the mapping defined by

$$\theta(17i+j) = j \ \ \forall \ 0 \le i \le 6, \ \forall \ 1 \le j \le 11$$

Since $\theta(\alpha_2) = (1, 2, \ldots, 17)$ and $\theta(\alpha_4) = (1, 9)(2, 6)(4, 5)(7, 8)$, then $K \cong \theta(K) = L_2(17)$. Let $H_0 = \langle \alpha_1, \alpha_3 \rangle$. Then $H_0 \cong L_2(13)$. Moreover, K conjugates H_0 into H_1, H_1 into H_2 and so it conjugates H_{10} into H_0,

where
$H_i = < (i, 17+i, 34+i, 51+i, 68+i, 85+i, 102+i, \ldots, 221+i)(i, 17+i)(34+i, 68+i) >$
$\forall \ 1 \le i \le 10$. Hence we get $L_2(13)wrL_2(17) \subseteq G$. On the other hand, Since $X = \alpha_1\alpha_2$ and $Y = \alpha_4\alpha_3^X$, then $G \subseteq L_2(13)wrL_2(17)$. Hence $G = L_2(13)wrL_2(17)$ ◊

Theorem: 3.2 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ can be generated using two permutations, the first is of order 252k and an involution, for all integers $k \ge 1$.

Proof: Let $\sigma = (1, 2, \ldots, 252k)$ and $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$. If $k=1$, then we get the group $L_2(13)wrL(17)$ which can be considered as the trivial wreath product $L_2(13)wrL(17)$ wr$<$id$>$. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{11} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{45} = (k, 2k, 3k, \ldots, 252k)$. Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i} \rangle$, be the groups acts on the sets $\Gamma_i = \{ i, k+i, 2k+i, \ldots, 251k+i \}$, for all $1 \le i \le k$. Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product $G_1 \times G_2 \times \ldots \times G_k$, where, by theorem 3.1 each $G_i \cong L_2(13)wrL_2(17)$. Let $\beta = \delta^{-1}\sigma = (1, 2, \ldots, k)(k+1, k+2, \ldots, 2k) \ldots (76k+1, 76k+2, \ldots, 252k)$. Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into $G_3, \ldots$ and G_k into G_1. Hence we get the wreath product $(L_2(13)wrL(17))wrC_K \subseteq G$. On the other hand, since $\delta\beta = (1, 2, \ldots, k, k+1, k+2, \ldots, 2k, \ldots, 251k+1, 251k+2, \ldots, 252k) = \sigma$, then $\sigma \in (L_2(13)wrL(17))wrC_K$.

Hence $G = < \sigma, \tau > \cong (L_2(13)wrL_2(17))wrC_K$.◊

Theorem: 3.3 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,S_k$ can be generated using three permutations, the first is of order 252k, the second and the third are involutions, for all $k \ge 2$.

Proof: Let $\sigma = (1, 2, \ldots, 252k)$, $\tau = (k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k)$ and $\mu = (1, 2)(k+1, k+2)(2k+1, 2k+2) \ldots (251k+1, 251k+2)$. Since by Theorem 3.2, $< \sigma, \tau > = (L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and $(1, 2, \ldots, k)(k+1, k+2, \ldots, 2k) \ldots (251k+1, \ldots, 252k) \in (L_2(13)\,wr\,L_2(17))\,wr\,C_k$ then $\langle (1,\ldots,k)(k+1,\ldots,2k)\ldots(251k+1,\ldots,252k), \mu \rangle \cong S_k$.

Hence $G = \langle \sigma, \tau, \mu \rangle \cong (L_2(13)wrL_2(17))wrS_k$. ◊

Corollary: 3.4 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,A_k$ can be generated using three permutations, the first is of order 252k, the second is an involution and the third is of order 3, for all odd integers- $k \ge 3$.

Theorem: 3.5 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,(S_m\,wr\,C_a)$ can be generated using three permutations, the first is of order 252k, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

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Proof: Let σ = (1, 2, …, 252k), τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k) and μ = (k, a)(2k, k+a)(3k, 2k+a) … (252k, 251k+a). Since by Theorem 3.2, $< \sigma, \tau > \cong (L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and (1, …, k)(k+1, …, 2k) … (251k+1, … ,252k)$\in$ $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ then $\langle$(1, …, k)(k+1, …, 2k) …(251k+1, …, 252k, μ $\rangle \cong (S_m\mathrm{wr}\,C_a)$.

Hence $G = \langle \sigma,\ \tau,\ \mu \rangle \cong (L_2(13)wrL_2(17))wr(S_mwrC_a)$.$\Diamond$

Theorem: 3.6 S_{252k+1} and A_{252k+1} can be generated using the wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and a transposition in S_{252k+1} for all integers $k > 1$ and an element of order 3 in A_{252k+1} for all odd integers $k > 1$.

Proof: Let σ = (1, 2, …, 252k), τ =(k, 9k)(2k, 6k)(4k, 5k)(7k, 8k)(12k, 20k, 23k, 31k)(13k, 17k)(15k, 16k)(18k, 19k)(24k, 28k)(26k, 27k)(29k, 30k)(34k, 42k, 56k, 64k)(35k, 39k)(37k, 38k)(40k, 41k)(45k, 53k)(46k, 50k)(48k, 49k)(51k, 52k)(57k, 61k)(59k, 60k)(62k, 63k)(67k, 75k)(68k, 72k)(70k, 71k), μ =(252k+1,1) and μ' =(1,k, 252k+1) be four permutations, of order 252k, 2, 2 and 3 respectively. Let $H = \langle \sigma,\ \tau \rangle$. By theorem 3.2 $H \cong (L_2(13)\,wr\,L_2(17))\,wr\,C_k$.

Case: 1 Let $G = \langle \sigma,\ \tau,\ \mu \rangle$. Let $\alpha = \sigma\mu$, then $\alpha = (1,2,...,252k,252k+1)$ which is a cycle of order $252k+1$. By theorem 2.4 $G < \sigma,\tau,\mu' > \cong < \alpha,\mu > \cong S_{252k+1}$.

Case: 2 Let $G = \langle \sigma,\ \tau,\mu' \rangle$. By theorem 2.5 $< \sigma,\mu' > \cong A_{252K+1}$. Since τ is an even permutation, then $G \cong A_{252K+1}$.

Theorem: 3.7 S_{252k+1} and A_{252K+1} can be generated using the wreath product $L_2(13)\,wr\,L_2(17)$ and an element of order $k + 1$ in S_{252k+1} and A_{252K+1} for all integers $k \geq 1$.

Proof: Let G=$\langle \sigma,\ \tau,\ \mu \rangle$, where, σ =(1, 2, 3, …, 252)(252(k-(k-1))+1, …, 252(k-(k-1))+252) … (252(k-1)+1, …, 252(k-1)+252), τ =(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) … (252(k-1)+1, 252(k-1)+9) … (252(k-1)+73, 252(k-1)+74), and μ =(252, 154, …,252k, 252k+1), where k-i >0, be three permutations of order 252, 4 and k+1 respectively. Let $H = \langle \sigma,\ \tau \rangle$. Define the mapping θ as follows;

$$\theta_{(17(k- i) +j)} = j \,\forall\, 1 \leq i \leq k \ , \ \forall\ 1 \leq j \leq 11$$

Hence $H =< \sigma,\tau > \cong L_2(13)\,wr\,L_2(17)$. Let $\alpha = \mu\sigma$ it is easy to show that $\alpha = (1,2,3,...,252k+1)$, which is a cycle of order $252k+1$. Let $\mu' = \mu^\sigma = (1,253,...,252(k-1)+1,252k+1)$ and $\beta = [\mu,\ \mu'] = (1,252,252k+1)$. Since $h.c.f\,(1,252,252k+1) = 1$, then by theorem 2.3 $G = \langle \sigma,\tau,\mu \rangle \cong \langle \alpha,\beta \rangle\,S_{252k+1}$ or A_{252K+1} depending on whether k is an odd or an even integer respectively. $\Diamond$

REFERENCES:

[1] A. M. Hammas and I. R. Al-Amri, Symmetric generating set of the alternating groups A_{2n+1}, JKAU: Educ. Sci., 7 (1994), 3-7.

[2] B. H. Shafee, Symmetric generating set of the groups A_{kn+1} and S_{kn+1} using th the wreath product A_m wr S_a, Far East Journal of Math. Sci. (FJMS), 28(3) (2008), 707-711.

[3] H. A. AL-Shehri and I. A. Al-Amri, Symmetric and permutational generating sets of S_{9k+1} and A_{9k+1} using the wreath product M_9wrC_k, Far East J. Mathe. Sci. (FJMS), 31(2) (2008), 227-235.

[4] I.R. Al-Amri, Computational methods in permutation groups, ph. D Thesis, University of St. Andrews, September 1992.

[5] I.R. Al-Amri, and A.M. Hammas, Symmetric generating set of Groups A_{kn+1} and S_{kn+1}, JKAU: Sci., 7 (1995), 111-115.

[6] J. H. Conway and others, Atlas of Finite Groups, Oxford Univ. Press, New York, 1985.

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