Research Journal of Pure Algebra -1(9), 2011, Page: 234-238
Available online through www.rjpa.info **ISSN 2248–9037**

On the Structure of Some Groups Containing $L_2(13)\, wr\, L_2(17)$

Basmah H Shafee*

Department of Mathematics, Umm Al qura University, Makkah, Saudi Arabia

E-mail: dr.basmah_1391@hotmail.com

(Received on: 19-11-11; Accepted on: 09-12-11)

ABSTRACT

In this paper, we will generate the wreath product $L_2(13)\, wr\, L_2(17)$ *using only two permutations. Also, we will show the structure of some groups containing the wreath product* $L_2(13)\, wr\, L_2(17)$. *The structure of the groups founded is determined in terms of wreath product* $(L_2(13)\, wr\, L_2(17))\, wr\, C_k$. *Some related cases are also included. Also, we will show that* S_{252k+1} *and* A_{252K+1} *can be generated using the wreath product* $(L_2(13)\, wr\, L_2(17))\, wr\, C_k$ *and a transposition in* S_{252k+1} *and an element of order* 3 *in* A_{252K+1}. *We will also show that* S_{252k+1} *and* A_{252K+1} *can be generated using the wreath product* $L_2(13)\, wr\, L_2(17)$ *and an element of order* $k+1$.

Keywords and phrases: *wreath product, Linear group.*

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1. INTRODUCTION:

Hammas and Al-Amri [1], have shown that A_{2n+1} of degree $2n+1$ can be generated using a copy of S_n and an element of order 3 in A_{2n+1}. They also gave the symmetric generating set of Groups A_{kn+1} and S_{kn+1} using S_n [5]
.

Shafee [2] showed that the groups A_{kn+1} and S_{kn+1} can be generated using the wreath product A_m wr S_a and an element of order *k*+1. Also she showed how to generate S_{kn+1} and A_{kn+1} symmetrically using *n* elements each of order *k*+1.

Al-Amri and Al-Shehri [3] have shown that S_{9k+1} and A_{9k+1} can be generated using the wreath product $M_9 \mathrm{wr} C_k$ and an element of order 4 in S_{9k+1} and element of order 5 in A_{9k+1}.

The Linear groups $L_2(13)$ and $L_2(17)$ are two groups of the well known simple groups. In [6], they are fully described. In a matter of fact, they can be faintly presented in different ways. They have presentations in [6] as follows
:

$$L_2(13) =< X,Y \mid X^{13} = Y^2 = (X^4YX^7Y)^2 = (XY)^3 = 1 >$$
$$L_2(17) =< X,Y \mid X^{17} = Y^2 = (X^4YX^9Y)^2 = (XY)^3 = 1 >$$

$L_2(13)$ can be generated using two permutations, the first is of order 13 and an involution as follows: $L_2(13) =< (1,2,..,13)(1,5)(3,4)(6,8)(7,14)(9,13)(10,11) >$.

$L_2(17)$ can be generated using two permutations, the first is of order 17 and an involution as follow: $L_2(17) =< (1,2,...,17)(1,16)(2,8)(3,11)(5,10)(6,14)(7,12)(9,15)(17,18) >$

.

In this paper, we will generate the wreath product $L_2(13)\, wr\, L_2(17)$ using only two permutations. Also, we show the structure of some groups containing the wreath product $L_2(13)\, wr\, L_2(17)$. The structure of the groups founded is

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***Corresponding author: Basmah H Shafee*, *E-mail: dr.basmah_1391@hotmail.com**

determined in terms of wreath product $(L_2(13)\, wr\, L_2(17))\, wr\, C_k$. Some related cases are also included. Also, we will show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $(L_2(13)\, wr\, L_2(17))\, wr\, C_k$ and a transposition in S_{252k+1} and an element of order 3 in A_{252K+1}. We will also show that S_{252k+1} and A_{252K+1} can be generated using the wreath product $L_2(13)\, wr\, L_2(17)$ and an element of order $k+1$.

2. PRELIMINARY RESULTS:

Definition: 2.1 Let *A* and *B* be groups of permutations on non empty sets Ω_1 and Ω_2 respectively. The wreath product of *A* and *B* is denote by *A* wr *B* and defined as *A* wr *B*= $A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of *A* and a mapping θ where $\theta: B \to \text{Aut}\,(A^{\Omega_2})$ is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that $|A \text{ wr } B| = (|A|)^{|\Omega_2|}\,|B|$.

Theorem: 2.2 [4] Let G be the group generated by the *n*-cycle (1, 2… *n*) and the 2-cycle (*n*, *a*). If $1 < a < n$ is an integer with $n = am$, then $G \cong S_m \text{ wr } C_a$.

Theorem 2.3 [4] Let $1 \le a \ne b < n$ be any integers. Let *n* be an odd integer and let G be the group generated by the *n*-cycle (1,2,…,*n*) and the 3-cycle (n,a,b). If the $hcf(n,a,b)$=1**,** then $G = A_n$. While if *n* can be an even then $G = S_n$.

Theorem: 2.4 [4] Let $1 \le a < n$ be any integer. Let $G = \langle\, (1,2,...,n)\,,\,(n,a)\,\rangle$. If $h.c.f.\,(n,a) = 1$, then $G = S_n$.

Theorem: 2.5 [4] Let $1 \le a \ne b < n$ be any integers. Let n be an even integer and let G be the group generated by the (*n*-1)-cycle $(1, 2, ..., n-1)$ and 3-cycle (*n*,*a*,*b*). Then $G = A_n$.

3. THE RESULTS:

Theorem: 3.1 The wreath product $L_2(13)\, wr\, L_2(17)$ can be generated using two permutations, the first is of order 252 and the second is of order 4.

Proof: Let *G* = ⟨*X*,*Y*⟩, where: *X*=(1, 2, 3, 4, …, 252), which is a cycle of order 252, *Y*=(1, 9)(2, 6)(4, 5)(7, 8)
(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27) (29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)
(46, 50)(48, 49) (51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74), which is the product of two cycles each of order 4 and twenty four transpositions. Let $\alpha_1 = \left((XY)^6[X,Y]^5\right)^{18}$. Then

$$\alpha_1 = (17, 22, 33, 44, 55, 66, 252),$$

which is a cycle of order 7. Let $\alpha_2 = \alpha_1^{-1}X$. It is easy to show that

$$\alpha_2 = (1, 2, 3\ldots 17)\ (18, 19, 20\ldots 22)\ \ldots\ (67, 68, 69\ldots 252),$$

which is the product of seven cycles each of order 17. Let: $\beta_1 = (Y^2)^{(XY)^{18}}$ =(9, 20)(12, 23)(31, 53)(34, 56), $\beta_2 = \beta_1 Y^{-1}$ =(1, 9, 12, 20)(2, 6)(4, 5) (7, 8)(13, 17)(15, 16)(18, 19)(23, 31, 45, 53)(24, 28)(26, 27)(29, 30)(34, 42)(35, 39)(37, 38)(40, 41)(46, 50)(48, 49)(51, 52)(56, 64)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71) (73, 74), $\beta_3 = (Y^3\beta_2)^2$ =(1, 45)(12, 23), $\beta_4 = \beta_3^{(\alpha_2^{-1}\alpha_1^3)}$ =(11, 44)(55, 66) and $\beta_5 = \beta_4^{\beta_3^{\alpha_2^{-1}}}$ = (17, 221)(68, 85). Let $\alpha_3 = \beta_5^{\beta_3^{(\alpha_2^{-1}\alpha_1)}}$.

Hence

$$\alpha_3 = (17, 34)\ (51, 85).$$

Let $\alpha_4 = YX^{-1}\alpha_3^{-1}X$. We can conclude that

α_4 =(1,9)(2,6)(4,5)(7,8)(12,20)(13,17)(15,16)(18,19)(23,31)(24,28)(26,27)(29,30)(34,42)(35,39)(37,38)(40,41)(45,53)(46,50)(48,49)(51,52)(56,64)(57,61)(59,60)(62,63)(67,75)(68,72)(70,71)(73,74),

which is the product of twenty eight transpositions. Let $K = \langle \alpha_2 , \alpha_4 \rangle$. Let $\theta : K \rightarrow L_2(17)$ be the mapping defined by

$$\theta(17i+j) = j \quad \forall\ 0 \le i \le 6,\ \forall\ 1 \le j \le 11$$

Since $\theta(\alpha_2) =$(1, 2, …, 17) and $\theta(\alpha_4) =$(1, 9)(2, 6)(4, 5)(7, 8), then $K \cong \theta(K) = L_2(17)$. Let $H_0 = \langle \alpha_1 , \alpha_3 \rangle$. Then $H_0 \cong L_2(13)$. Moreover, *K* conjugates H_0 into H_1, H_1 into H_2 and so it conjugates H_{10} into H_0 ,

where

$H_i =< (i,17+i,34+i,51+i,68+i,85+i,102+i,....,221+i)(i,17+i)(34+i,68+i) >$

$\forall\ 1 \le i \le 10$. Hence we get $L_2(13)wrL_2(17) \subseteq G$. On the other hand, Since *X*= $\alpha_1\alpha_2$ and *Y*= $\alpha_4\alpha_3^{\,X}$, then $G \subseteq L_2(13)wrL_2(17)$. Hence $G = L_2(13)wrL_2(17)$ ◊

Theorem: 3.2 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ can be generated using two permutations, the first is of order 252*k* and an involution, for all integers *k* ≥ 1.

Proof: Let σ = (1, 2, …, 252*k*) and τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*) (59*k*, 60*k*) (62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*). If *k*=1, then we get the group $L_2(13)wrL(17)$ which can be considered as the trivial wreath product $L_2(13)wrL(17)$ wr<id>. Assume that $k > 1$. Let $\alpha = \prod_{i=0}^{11} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{45}$ =(*k*, 2*k*, 3*k*, …, 252*k*). Let G_i =⟨ δ^{σ^i} , τ^{σ^i} ⟩, be the groups acts on the sets Γ_i ={ i, *k*+i, 2*k*+i,…, 251*k*+*i* }, for all $1 \le i \le k$. Since $\bigcap_{i=1}^{k} \Gamma_i = \varphi$, then we get the direct product $G_1 \times G_2 \times \ldots \times G_k$, where, by theorem 3.1 each $G_i \cong L_2(13)wrL_2(17)$. Let $\beta = \delta^{-1}\sigma$ = (1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (76*k*+1, 76*k*+2, …, 252*k*). Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into G_3,…and G_k into G_1. Hence we get the wreath product $(L_2(13)wrL(17))wrC_K \subseteq G$. On the other hand, since $\delta\beta$ = (1, 2, …, *k*, *k*+1, *k*+2, …, 2*k*, …, 251*k*+1, 251*k*+2, …, 252*k*)= σ, then $\sigma \in (L_2(13)wrL(17))wrC_K$.

Hence $G =< \sigma, \tau > \cong (L_2(13)wrL_2(17))wrC_K$.◊

Theorem: 3.3 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,S_k$ can be generated using three permutations, the first is of order 252*k*, the second and the third are involutions, for all *k* ≥ 2.

Proof: Let σ = (1, 2, …, 252*k*), τ =(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ = (1, 2)(*k*+1, *k*+2)(2*k*+1, 2*k*+2) … (251*k*+1, 251*k*+2). Since by Theorem 3.2, $< \sigma, \tau >= (L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and (1, 2, …, *k*)(*k*+1, *k*+2, …, 2*k*) … (251*k*+1, …,252*k*)∈ $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ then ⟨(1,…,*k*)(*k*+1,…,2*k*)…(251*k*+1,…,252*k*) , μ ⟩ $\cong S_k$.

Hence $G = \langle \sigma,\ \tau,\ \mu \rangle \cong (L_2(13)wrL_2(17))wrS_k$. ◊

Corollary: 3.4 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,A_k$ can be generated using three permutations, the first is of order 252*k*, the second is an involution and the third is of order 3, for all odd integers *k* ≥ 3.

Theorem: 3.5 The wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,(S_m\ wr\ C_a)$ can be generated using three permutations, the first is of order 252*k*, the second and the third are involutions, where $k = am$ be any integer with $1 < a < k$.

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Proof: Let σ= (1, 2, …, 252*k*), τ=(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*) and μ = (*k*, *a*)(2*k*, *k*+*a*)(3*k*, 2*k*+*a*) … (252*k*, 251*k*+*a*). Since by Theorem 3.2, $<\sigma,\tau>\cong(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and (1, …, *k*)(*k*+1, …, 2*k*) … (251*k*+1, … ,252*k*)∈ $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ then ⟨(1, …, *k*)(*k*+1, …, 2*k*) …(251*k*+1, …, 252*k*, $\mu\rangle\cong(S_m\mathrm{wr}\,C_a)$.

Hence $G=\langle\sigma,\ \tau,\ \mu\rangle\cong(L_2(13)wrL_2(17))wr(S_mwrC_a)$.◊

Theorem: 3.6 S_{252k+1} and A_{252k+1} can be generated using the wreath product $(L_2(13)\,wr\,L_2(17))\,wr\,C_k$ and a transposition in S_{252k+1} for all integers $k>1$ and an element of order 3 in A_{252k+1} for all odd integers $k>1$.

Proof: Let σ= (1, 2, …, 252*k*), τ=(*k*, 9*k*)(2*k*, 6*k*)(4*k*, 5*k*)(7*k*, 8*k*)(12*k*, 20*k*, 23*k*, 31*k*)(13*k*, 17*k*)(15*k*, 16*k*)(18*k*, 19*k*)(24*k*, 28*k*)(26*k*, 27*k*)(29*k*, 30*k*)(34*k*, 42*k*, 56*k*, 64*k*)(35*k*, 39*k*)(37*k*, 38*k*)(40*k*, 41*k*)(45*k*, 53*k*)(46*k*, 50*k*)(48*k*, 49*k*)(51*k*, 52*k*)(57*k*, 61*k*)(59*k*, 60*k*)(62*k*, 63*k*)(67*k*, 75*k*)(68*k*, 72*k*)(70*k*, 71*k*), μ =(252*k*+1,1) and μ'=(1,*k*, 252*k*+1) be four permutations, of order 252*k*, 2, 2 and 3 respectively. Let $H=\langle\sigma,\ \tau\rangle$. By theorem 3.2 $H\cong(L_2(13)\,wr\,L_2(17))\,wr\,C_k$.

Case: 1 Let $G=\langle\sigma,\ \tau,\ \mu\rangle$. Let $\alpha=\sigma\mu$, then $\alpha=(1,2,...,252k,252k+1)$ which is a cycle of order $252k+1$. By theorem 2.4 $G<\sigma,\tau,\mu'>\cong<\alpha,\mu>\cong\ S_{252k+1}$.

Case: 2 Let $G=\langle\sigma,\ \tau,\mu'\rangle$.By theorem 2.5 $<\sigma,\mu'>\cong A_{252K+1}$. Since τ is an even permutation, then $G\cong A_{252K+1}$.

Theorem: 3.7 S_{252k+1} and A_{252K+1} can be generated using the wreath product $L_2(13)\,wr\,L_2(17)$ and an element of order $k+1$ in S_{252k+1} and A_{252K+1} for all integers $k\geq1$.

Proof: Let *G*=$\langle\sigma,\ \tau\ ,\mu\rangle$, where, σ=(1, 2, 3, …, 252)(252(*k*-(*k*-1))+1, …, 252(*k*-(*k*-1))+252) … (252(*k*-1)+1, …, 252(*k*-1)+252), τ=(1, 9)(2, 6)(4, 5)(7, 8)(12, 20, 23, 31)(13, 17)(15, 16)(18, 19)(24, 28)(26, 27)(29, 30)(34, 42, 56, 64)(35, 39)(37, 38)(40, 41)(45, 53)(46, 50)(48, 49)(51, 52)(57, 61)(59, 60)(62, 63)(67, 75)(68, 72)(70, 71)(73, 74) … (252(*k*-1)+1, 252(*k*-1)+9) … (252(*k*-1)+73, 252(*k*-1)+74), and μ =(252, 154, …,252*k*, 252*k*+1), where *k*-*i* >0, be three permutations of order 252, 4 and *k*+1 respectively. Let $H=\langle\sigma,\ \tau\rangle$. Define the mapping θ as follows;

$$\theta(17(k-i)+j)=j\ \forall\ 1\leq i\leq k\ ,\ \forall\ 1\leq j\leq11$$

Hence $H=<\sigma,\tau>\cong L_2(13)\,wr\,L_2(17)$. Let $\alpha=\mu\sigma$ it is easy to show that $\alpha=(1,2,3,...,252k+1)$, which is a cycle of order $252k+1$. Let $\mu'=\mu^{\sigma}=(1,253,...,252(k-1)+1,252k+1)$ and $\beta=[\mu,\ \mu']=(1,252,252k+1)$. Since $h.c.f\,(1,252,252k+1)$ =1, then by theorem 2.3 $G=\langle\sigma,\tau,\mu\rangle\cong\langle\alpha,\beta\rangle\ S_{252k+1}$ or A_{252K+1} depending on whether *k* is an odd or an even integer respectively. ◊

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