

GENERAL RELATIONS FOR ABSOLUTE SUMMABILITY

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ABSTRACT

New general results concerning absolute summability of an infinite series are presented. Other special cases are also deduced.

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1. INTRODUCTION:

Let T be a lower triangular matrix, (s_n) a sequence, then

$$T_n := \sum_{v=0}^n t_{nv} s_v \tag{1}$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty \tag{2}$$

Given any lower triangular matrix T one can associate the matrices \bar{T} and T , with entries defined by,

$$t_{nv} = \sum_{i=v}^n t_{ni}, \quad n, i=0, 1, 2, \dots, \quad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1, v}$$

respectively. With $s_n = \sum_{i=0}^n a_i \lambda_i$,

$$t_n = \sum_{v=0}^n t_{nv} s_v = \sum_{v=0}^n t_{nv} \sum_{i=0}^v a_i \lambda_i = \sum_{i=0}^n a_i \lambda_i \sum_{v=i}^n t_{nv} = \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i \tag{3}$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1, i} a_i \lambda_i = \left| \sum_{i=0}^n \bar{t}_{ni} a_i \lambda_i \right| \text{ as } \bar{t}_{n-1, n} = 0 \tag{4}$$

$$X_n = u_n - u_{n-1} = \sum_{i=0}^n u_{ni} a_i. \tag{5}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n . A triangle A is called factorable if its nonzero entries a_{mn} can be written in the form $b_m c_n$ for each m and n . We also assume that $U = (u_{ij})$ is a triangle $(p_n), (q_n)$ are assumed to be positive sequences of numbers such that

$$P_n = p_0 + p_1 + \dots + p_n \rightarrow \infty, \text{ as } n \rightarrow \infty,$$

$$Q_n = q_0 + q_1 + \dots + q_n \rightarrow \infty, \text{ as } n \rightarrow \infty,$$

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The series $\sum a_n$ is said to be summable $|R, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta z_{n-1}|^k < \infty,$$

where

$$z_n = \sum_{i=0}^n p_i s_i.$$

Via giving another name of this kind of summability (that is replacing $|R, p_n|_k$ by $|\bar{N}, p_n|_k$ with exactly the same meaning, Savas [1] gave the following result

Theorem: 1.1 Let $1 < k \leq s < \infty$. Let (λ_n) be a sequence of constants, T be a triangle with bounded entries such that T and (p_n) satisfy

$$(i) \quad t_{vv} \lambda_v = O\left(\left(\frac{p_v}{P_v}\right) v^{1/s-1/k}\right).$$

$$(ii) \quad (n|X_n|)^{s-k} = O(1),$$

$$(iii) \quad \sum_{v=1}^{n-1} |\Delta_v(t_{nv} \lambda_v)| = O(|t_{nn} \lambda_n|),$$

$$(iv) \quad \sum_{n=v+1}^{\infty} (n|t_{nn} \lambda_n|)^{s-1} |\Delta_v(\hat{t}_{nv} \lambda_v)| = O(v^{s-1} |t_{vv} \lambda_v|^s),$$

$$(v) \quad \sum_{v=1}^{n-1} |t_{vv} \lambda_v| |\hat{t}_{nv} \lambda_v| = O(|t_{nn} \lambda_n|), \text{ and}$$

$$(vi) \quad \sum_{n=v+1}^{\infty} (n|t_{nn} \lambda_n|)^{s-1} |\hat{t}_{nv} \lambda_v| = O\left((v|t_{vv} \lambda_v|)^{s-1}\right),$$

$$\text{where } \left(X_n = u_n - u_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v, u_n = \frac{1}{P_n} \sum_{i=0}^n p_i s_i \right)$$

Then the series $\sum a_n \lambda_n$ is summable $|T|_k$ whenever $\sum a_n$ is summable $|\bar{N}, p_n|_k$.

2. RESULT:

The aim of this paper is to present the following general result:

Theorem: 2.1 Let $1 < k \leq s < \infty, (\lambda_n)$ be a sequence of constants. Let T and U be triangles with bounded entries such that

U is factorable, that is u_{nv} can be written as $u_{nv} = \phi_n \phi_v$, and they satisfy the following:

$$(i) \quad t_{vv} \lambda_v = O\left(v^{1/s-1/k} |\phi_v \phi_{v+1}|\right),$$

$$(ii) \quad (n|X_n|)^{s-k} = O(1),$$

$$(iii) \quad \sum_{v=1}^{n-1} |\Delta_v(\hat{t}_{nv} \lambda_v)| = O(|t_{nn} \lambda_n|),$$

$$(iv) \quad \sum_{n=v+1}^{\infty} (n|t_{nn} \lambda_n|)^{s-1} |\Delta_v(\hat{t}_{nv} \lambda_v)| = O\left(v^{s-1} |t_{vv} \lambda_v|^s\right),$$

- (v) $\sum_{v=1}^{n-1} |t_{vv}\lambda_v| |\hat{t}_{nv}\lambda_v| = O(|t_{nn}\lambda_n|),$
- (vi) $\sum_{n=v+1}^{\infty} (n|t_{nn}\lambda_n|)^{s-1} |\hat{t}_{nv}\lambda_v| = O\left((v|t_{vv}\lambda_v|)^{s-1}\right),$ and
- (vii) $\Delta\phi_v^{-1} = O(|\phi_v|).$

Then the series $\sum a_n \lambda_n$ is summable $|T|_s$ whenever $\sum a_n$ is summable $|U|_k$. It may be mentioned that theorem 1.1 remains a special case of theorem 2.1 in the sense that

If we are putting $U \equiv (R, p_n)$, that is $U = (u_{nv})$, where $\hat{u}_{nv} = \phi_n \phi_v$, $\phi_n = \frac{p_n}{P_n P_{n-1}}$, $\phi_v = p_{v-1}$ in theorem 2.1 we obtain theorem 1.1.

Proof: By Abel’s transformation we have

$$\begin{aligned} Y_n &= \sum_{v=1}^n \phi_v a_v \frac{\hat{t}_{nv}\lambda_v}{\phi_v} \\ &= \sum_{v=1}^{n-1} \left(\sum_{r=1}^v \phi_r a_r \right) \Delta_v \left(\frac{\hat{t}_{nv}\lambda_v}{\phi_v} \right) + \left(\sum_{v=1}^n \phi_v a_v \right) \frac{t_{nv}\lambda_n}{\phi_n} \\ &= \sum_{v=1}^{n-1} \frac{X_v}{\phi_v} \left(\Delta\phi^{-1} t_{nv}\lambda_v + \phi_{v+1}^{-1} \Delta_v (\hat{t}_{nv}\lambda_v) \right) + \frac{X_n t_{nn}\lambda_n}{\phi_n \phi_n} \\ &= Y_{n1} + Y_{n2} + Y_{n3}. \end{aligned}$$

To complete the proof, by Minkowski’s inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1} |Y_{nj}|^s < \infty, \quad j = 1, 2, 3.$$

Now applying Hölder’s inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{s-1} |Y_{n1}|^s &= \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{v=1}^{n-1} \Delta(\phi_v^{-1}) \frac{X_v \hat{t}_{nv}\lambda_v}{\phi_v} |t_{vv}\lambda_v|^{-1} |t_{vv}\lambda_v| \right|^s \\ &\leq \sum_{n=1}^{\infty} n^{s-1} \sum_{v=1}^{n-1} \left| \Delta(\phi_v^{-1}) \right|^s \frac{|\hat{t}_{nv}\lambda_v| |X_v|^s |t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s} \left(\sum_{v=0}^{n-1} |t_{vv}\lambda_v| |\hat{t}_{nv}\lambda_v| \right)^{s-1} \\ &= O(1) \sum_{n=1}^{\infty} n^{s-1} |t_{nn}\lambda_n|^{s-1} \sum_{v=1}^{n-1} \left| \Delta(\phi_v^{-1}) \right|^s \frac{|X_v|^s |t_{vv}\lambda_v|^{1-s} |\hat{t}_{nv}\lambda_v|}{|\phi_v|^s} \\ &= O(1) \sum_{v=1}^{\infty} \left| \Delta(\phi_v^{-1}) \right|^s \frac{|X_v|^s |t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s} \sum_{n=v+1}^{\infty} n^{s-1} |t_{nn}\lambda_n|^{s-1} |\hat{t}_{nv}\lambda_n| \\ &= O(1) \sum_{v=1}^{\infty} \left| \Delta(\phi_v^{-1}) \right|^s \frac{|X_v|^s |t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s} (v|t_{vv}\lambda_v|)^{s-1} \\ &= O(1) \sum_{v=1}^{\infty} v^{s-1} |X_v|^s \end{aligned}$$

$$\begin{aligned}
 &= O(1) \sum_{v=1}^{\infty} v^{k-1} |X_v|^k (v|X_v|)^{s-k} \\
 &= O(1) \sum_{v=1}^{\infty} v^{s-1} |X_v|^k = O(1). \\
 \sum_{n=1}^{\infty} n^{s-1} |Y_{n2}|^s &= \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{v=1}^{n-1} \frac{\Delta_v(t_{nv}\lambda_v)}{\phi_v \varphi_{v+1}} X_v \right|^s \\
 &\leq \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{v=1}^{n-1} \frac{|\Delta_v(t_{nv}\lambda_v)|}{|\phi_v| |\varphi_{v+1}|} |X_v| \right)^s \\
 &\leq \sum_{n=1}^{\infty} n^{s-1} \sum_{v=1}^{n-1} \frac{|\Delta_v(t_{nv}\lambda_v)| |X_v|^s}{|\phi_v|^s |\varphi_{v+1}|^s} \left(\sum_{v=1}^{n-1} |\Delta_v(t_{nv}\lambda_v)| \right)^{s-1} \\
 &= O(1) \sum_{n=1}^{\infty} n^{s-1} |t_{nn}\lambda_n|^{s-1} \sum_{v=0}^{n-1} \frac{|\Delta_v(t_{nv}\lambda_v)| |X_v|^s}{|\phi_v|^s |\varphi_{v+1}|^s} \\
 &= O(1) \sum_{v=1}^{\infty} \frac{|X_v|^s}{|\phi_v|^s |\varphi_{v+1}|^s} \sum_{n=v+1}^{\infty} n^{s-1} |t_{nn}\lambda_n|^{s-1} |\Delta_v(t_{nv}\lambda_v)| \\
 &= O(1) \sum_{v=1}^{\infty} \frac{v^{s-1} |t_{vv}\lambda_v|^s |X_v|^s}{|\phi_v|^s |\varphi_{v+1}|^s} \\
 &= O(1) \sum_{v=1}^{\infty} v^{s-s/k} |X_v|^s \\
 &= O(1) \sum_{v=1}^{\infty} v^{k-1} |X_v|^k (v^{s-s/k-k+1} |X_v|^{s-k}) \\
 &= O(1) \sum_{v=1}^{\infty} v^{k-1} |X_v|^k = O(1)
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=1}^{\infty} n^{s-1} |Y_{n3}|^s &= \sum_{n=1}^{\infty} n^{s-1} \left| \frac{t_{nn} X_n \lambda_n}{\phi_n \varphi_n} \right|^s \\
 &\leq \sum_{n=1}^{\infty} n^{s-1} \frac{|t_{nn}\lambda_n|^s |X_n|^s}{|\phi_n|^s |\varphi_n|^s} \\
 &= O(1) \sum_{n=1}^{\infty} n^{s-1} |X_n|^s \\
 &= O(1) \sum_{n=1}^{\infty} n^{k-1} |X_n|^k = O(1).
 \end{aligned}$$

Theorem: 2.2 Let $1 < k < \infty$, (λ_n) be a sequence of constants. Let T and U be triangles with bounded entries such that \hat{U} is factorable, that is \hat{u}_{nv} can be written as $\hat{u}_{nv} = \phi_n \varphi_v$, and they satisfy the following:

(i) $t_{vv}\lambda_v = O(|\phi_v \varphi_{v+1}|)$.

(ii) $\sum_{v=1}^{n-1} |\Delta_v(\hat{t}_{nv}\lambda_v)| = O(|t_{nn}\lambda_n|)$,

- (iii) $\sum_{n=v+1}^{\infty} (n|t_{nn}\lambda_n|)^{k-1} |\Delta(\hat{t}_{nv}\lambda_v)| = O\left(v^{k-1}|t_{vv}\lambda_v|^k\right)$
- (iv) $\sum_{v=1}^{n-1} |t_{vv}\lambda_v| |\hat{t}_{nv}\lambda_v| = O(|t_{nn}\lambda_n|)$, and
- (v) $\sum_{n=v+1}^{\infty} (n|t_{nn}\lambda_n|)^{k-1} |\hat{t}_{nv}\lambda_v| = O\left((v|t_{vv}\lambda_v|)\right)$, and
- (vi) $\Delta\phi_v^{-1} = O(|\phi_v|)$.

Then the series $\sum a_n \lambda_n$ is summable $|T|_k$ whenever $\sum a_n$ is summable $|U|_k$.

Proof: Follows from Theorem 2.1 by putting $s = k$.

Corollary: 2.3 Sufficient conditions for the series $\sum a_n$ to be summable $|R, q_n|_k$, $k > 1$, whenever it is summable $|R, p_n|_k$ are

- (i) $q_n P_n = O(p_n Q_n)$,
- (ii) $\sum_{n=v+1}^{\infty} \frac{n^{s-1} q_n^s}{Q^s Q_{n-1}} = O\left(\frac{v^{s-1} q_v^{s-1}}{Q_v^s}\right)$.

Proof: Follows from theorem 2.2.by putting.

$$\hat{t}_{nv} = \frac{q_n Q_{v-1}}{Q_n Q_{n-1}}, \hat{u}_{nv} = \frac{p_n P_{v-1}}{P_n P_{n-1}}, \phi_n = \frac{p_n}{P_n P_{n-1}}, \phi_v = P_{v-1}, \text{ and } \lambda_n = 1,$$

noticing that $\hat{t}_{vv} = t_{vv}$, $\hat{u}_{vv} = u_{vv}$ for all v. We will show that conditions of Theorem 2.2 are all satisfied.

- (1) By (i) $t_{vv} = \frac{q_v}{Q_v} = O\left(\frac{p_v}{P_v}\right) = O\left(\frac{p_v P_v}{P_v P_{v-1}}\right) = O(|\phi_v \phi_{v+1}|)$.
- (2) $\sum_{v=1}^{n-1} |\Delta_v \hat{t}_{nv}| = \sum_{v=1}^{n-1} \frac{q_n q_v}{Q_n Q_{n-1}} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} q_v = \frac{q_n}{Q_n} = O(|t_{nn}|)$.
- (3) By (ii) $\sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} |\Delta_v \hat{t}_{nv}| = \sum_{n=v+1}^{\infty} \left(\frac{n q_n}{Q_n}\right)^{k-1} \frac{q_n q_v}{Q_n Q_{n-1}} = q_v \sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}}$
 $= O\left(\frac{v^{k-1} q_v^k}{Q_v^k}\right) = O\left(v^{k-1} |t_{vv}|^k\right)$.
- (4) $\sum_{v=1}^{n-1} |t_{vv}| |\hat{t}_{nv}| = \sum_{v=1}^{n-1} \frac{q_v}{Q_v} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} = O\left(\frac{q_n}{Q_n Q_{n-1}}\right) \sum_{v=1}^{n-1} q_v = O\left(\frac{q_n}{Q_n}\right) = O(|t_{nn}|)$.
- (5) $\sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} |\hat{t}_{nv}| = \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n}\right)^{k-1} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} = Q_{v-1} \left|\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}}\right|$
 $= O\left(\frac{v^{k-1} q_v^{k-1}}{Q_v^{k-1}}\right), \text{ by (ii)}$
 $= O\left((v|t_{vv}|)^{k-1}\right)$.

Corollary: 2.4 The series $\sum a_n$ is summable $|R, q_n|_k$ iff it is summable $|R, p_n|_k$, $k > 1$, provided the following holds

- (i) $q_n P_n = O(p_n Q_n)$,
- (ii) $p_n Q_n = O(q_n P_n)$,
- (iii) $\sum_{n=v+1}^{\infty} \frac{n^{s-1} q_n^s}{Q_n^s Q_{n-1}^s} = O\left(\frac{v^{s-1} q_v^{s-1}}{Q_v^s}\right)$,
- (iv) $\sum_{n=v+1}^{\infty} \frac{n^{s-1} p_n^s}{P_n^s P_{n-1}^s} = O\left(\frac{v^{s-1} p_v^{s-1}}{P_v^s}\right)$,

Proof: Follows from Corollary 2.3, via changing the rolls of $(p_n), (q_n)$.

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