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FIXED POINT THEOREMS ON COMPLETE G-CONE METRIC SPACE

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ABSTRACT

The aim of this paper is to discuss some fixed point theorems for contractive mappings in comple G- cone metric space.

Key Words and Phrases: *G- cone metric space, Fixed points.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

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1. INTRODUCTION:

It is well known that Banach contraction principle is a fundamental result in fixed point theory. Z. Mustafa, and B. Sims [8] introduced an appropriate generalization of metric space and obtained fixed point theorems for different contractive mappings in G- metric space. Huang and Zhang [5] introduced the concept of cone metric space. The results in [5] where generalized by Sh. Rezapour and R. Hamlbarani [9] by omitting the normality condition, which is a mile stone in developing fixed point theory in cone metric space. Ismat Beg. Mujahid Abbas and Talat Nazir [6] introduced the concept of G – cone metric space by replacing the set of real numbers by ordered Banach space. G- cone metric space is more general than that of a G – metric space and cone metric space. Here we recall some definition and results in [5] and [6].

2. PRELIMINARIES:

Definition: 2.1 Let E be a real Banach space. A subset $P \subseteq E$ is said to be a cone if and only if

1. P is closed, nonempty and $P \neq \{0\}$
2. $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$
3. $P \cap (-P) = \{0\}$

For a given cone P subset of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y – x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y – x \in intP$ where intP denotes interior of P.

Here we recall some definition and results in [6] which will be used in the theorems.

Definition: 2.2 Let X be a nonempty set. Suppose a mapping $G: X \times X \times X \to E$ satisfies

1. $G(x, y, z) = 0$ if and only if $x = y = z$.
2. $0 < G(x, y, z)$; whenever $x \neq y$, for all $x, y \in X$
3. $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots\ldots\ldots\ldots$.(symmetric in all the three variables.)
4. $G(x, x, y) \leq G(x, y, z)$; whenever $y \neq z$.
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$

Then G is called a generalized cone metric on X, is called a generalized cone metric space or G – cone metric space.

The idea of a G – cone metric space is more than that of a cone metric space.

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Definition: 2.3 Let X be a G – cone metric space $\{x_n\}$ be a sequence in X

(1) $\{x_n\}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a positive integer N such that for all n, m, l > N, $G(x_n, x_m, x_l) \leq c$

(2) $\{x_n\}$ is said to be a convergent sequence if for every $c \in E$ with $0 \ll c$, there is a positive integer N such that for all m, n > N, $G(x_m, x_n, x) \leq c$ for some fixed x in X.

A G – cone metric space X is said to be complete if every Cauchy sequence in X convergence.

Lemma: 2.1 Let X be a G – cone metric space $\{x_m\}$, $\{y_n\}$, $\{z_l\}$ be sequences in X such that $x_m \to x$, $y_n \to y$, $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$.

Lemma: 2.2 Let $\{x_n\}$ be a sequence in G – cone metric space X and $x \in X$. If $\{x_n\}$ converges to x, and $\{x_n\}$ converges to y, then x = y.

Lemma: 2.3 Let $\{x_n\}$ be a sequence in G – cone metric space X and $x \in X$. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Lemma: 2.4 Let $\{x_n\}$ be a sequence in a G – cone metric space X and if $\{x_n\}$ is a Cauchy sequence, then $G(x_m, x_n, x_l) \to 0$ as m, n, $l \to \infty$.

3. MAIN RESULTS:

Theorem: 3.1 Let T be a mapping on a complete G – cone metric space X into itself that satisfies

$$G(Tx, Ty, Ty) \leq k\,[G(x, Tx, Tx) \vee G(y, Ty, Ty)] \qquad (1)$$

For all $x, y \in X$, and $0 \leq k \leq 1$. Then T has a unique fixed point.

Proof: Let x_0 be an arbitrary point in X. Define $\{x_n\}$ in X such that $x_n = T^n x_0$. If $T^{n+1} x_0 = T^n x_0$ for some n, then T has a fixed point. Assume $T^{n+1} x_0 \neq T^n x_0$ for each n. By (1)

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\,[G(x_{n-1}, x_n, x_n) \vee G(x_n, x_{n+1}, x_{n+1})] \qquad (2)$$

If

$$G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1})$$

Then

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_n, x_{n+1}, x_{n+1})$$

That is

$$(1-k)\, G(x_n, x_{x+1}, x_{n+1}) \leq 0$$

Since 1- k > 0

$$G(x_n, x_{n+1}, x_{n+1}) = 0$$

implies $x_n = x_{n+1}$ which contradicts $x_n \neq x_{n+1}$ for each n. Therefore

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$$

Hence

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n) \text{ for every } n$$

Therefore

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n\, G(x_0, x_1, x_1)$$

Let n > m, then

$$G(x_m, x_n, x_n) \leq G(x_m, x_{m+1}, x_{m+1}) + \ldots\ldots\ldots + G(x_{n-1}, x_n, x_n)$$

$$\leq (k^m + k^{m+1} + \ldots\ldots\ldots + k^{n-1})\, G(x_0, x_1, x_1)$$

$$\leq \frac{k^m}{1-k} G(x_0, x_1, x_1)$$

Let c > 0, then there is a $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y \in E : \| y \| < \delta\}$. Since k < 1 there is a positive integer N such that $\left\| \frac{k^m}{1-k} G(x_0, x_1, x_1) \right\| \leq \delta$ for every $m \geq N$.

Therefore $\frac{k^m}{1-k} G(x_0, x_1, x_1) \in N_\delta(0)$

Hence $-\frac{k^m}{1-k} G(x_0, x_1, x_1) \in N_\delta(0)$

Therefore $c - \frac{k^m}{1-k} G(x_0, x_1, x_1) \in c + N_\delta(0) \subseteq P$

That is $\frac{k^m}{1-k} G(x_0, x_1, x_1) \leq c$ for $m \geq N$

Hence by (3) $G(x_n, x_m, x_l) \leq c$ $n, m \geq N$

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete there is an $z \in X$ such that $\{x_n\}$ converges to z.

Now we shall prove z is a fixed point of T. We have

$$G(x_{n+1}, Tz, Tz) \leq k\{G(x_n, x_{n+1}, x_{n+1}) \vee G(z, Tz, Tz)\}$$

If $G(x_n, x_{n+1}, x_{n+1}) \leq G(z, Tz, Tz)$

Then $G(x_{n+1}, Tz, Tz) \leq k\, G(z, Tz, Tz)$

Letting $n \to \infty$, we get

$$G(z, Tz, Tz) \leq k\, G(z, Tz, Tz)$$

That is $(1-k)\, G(z, Tz, Tz) \leq 0$

But 1 – k > 0. Therefore G (z, Tz, Tz) = 0. Hence Tz = z. If

$$G(z, Tz, Tz) \leq G(x_n, x_{n+1}, x_{n+1})$$

Then $G(x_{n+1}, Tz, Tz) \leq k\, G(x_n, x_{n+1}, x_{n+1})$

Letting $n \to \infty$ we get

$$G(z, Tz, Tz) \leq k\, G(z, z, z) = 0$$

which implies G (z, Tz, Tz) = 0, hence Tz = z. Therefore z is a fixed point of T.

Next we shall prove the fixed point is unique. Let 'z'' be another fixed point of T. So $Tz' = z'$. We have

$$\begin{aligned} G(z, z', z') &= G(Tz, Tz', Tz') \\ &\leq k\{G(z, Tz, Tz) \vee G(z', Tz', Tz')\} \\ &= k\{G(z, z, z) \vee G(z', z', z')\} \\ G(z, z', z') &= 0 \end{aligned}$$

Which implies $z = z'$. Hence the theorem.

Theorem: 3.2 Let (X, G) be a complete G – cone metric space, and $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Ty) \le k\{G(x, Ty, Ty) \vee G(y, Tx, Tx) \vee G(y, Ty, Ty)\} \tag{4}$$

For all $x, y \in X$, $k \in [0, ½]$. Then T has a unique fixed point.

Proof: $G(Tx, Ty, Ty) \le \{G(x, Ty, Ty) \vee G(y, Tx, Tx) \vee G(y, Ty, Ty)\}$

Therefore

$$G(Tx, Ty, Ty) \le k\,G(x, Ty, Ty) \tag{5}$$

Or $$G(Tx, Ty, Ty) \le k\,G(y, Tx, Tx) \tag{6}$$

Or $$G(Tx, Ty, Ty) \le k\,G(y, Ty, Ty) \tag{7}$$

Let x_0 be an arbitrary element in X. Define a sequence $\{x_n\}$ in X as follows $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, ………………… $x_n = Tx_{n-1} = T^nx_0$. If $x_n = x_{n+1}$ for some n, then T has a fixed point. Assume $x_n \ne x_{n+1}$ for each n. By (5).

$$G(x_n, x_{n+1}, x_{n+1}) \le k\,G(x_{n-1}, x_{n+1}, x_{n+1})$$

We have

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})$$

Therefore

$$G(x_n, x_{n+1}, x_{n+1}) \le k\,G(x_{n-1}, x_n, x_n) + k\,G(x_n, x_{n+1}, x_{n+1})$$

That is

$$(1-k)\,G(x_n, x_{n+1}, x_{n+1}) \le k\,G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{k}{1-k}\,G(x_{n-1}, x_n, x_n)$$

Let $p = \frac{k}{1-k} < 1$ since $k < ½$. Hence

$$G(x_n, x_{n+1}, x_{n+1}) \le p\,G(x_{n-1}, x_n, x_n) \text{ for every } n$$

So

$$G(x_n, x_{n+1}, x_{n+1}) \le p^n\,G(x_0, x_1, x_1)$$

Let n >m

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\le G(x_m, x_{m+1}, x_{m+1}) + \dots\dots\dots + G(x_{n+1}, x_n, x_n) \\ &\le (p^m + p^{m+1} + \dots\dots\dots + p^{n+1})\,G(x_0, x_1, x_1) \\ &\le \frac{p^m}{1-p}\,G(x_0, x_1, x_1) \end{aligned} \tag{8}$$

Let $c \ge 0$, then there is a $\delta > 0$ such that $c + N_\delta(0) \subseteq P$. Since $P < 1$ for $\delta > 0$, there is a positive integer N such that $\left\|\frac{p^m}{1-p}G(x_0, x_1, x_1)\right\| < \delta$ for $m \ge N$. Hence $\left\|-\frac{p^m}{1-p}G(x_0, x_1, x_1)\right\| < \delta$ for $m \ge N$.

Therefore $c - \frac{p^m}{1-p}G(x_0, x_1, x_1) \in P$. That is $\frac{p^m}{1-p}G(x_0, x_1, x_1) \le c$. By (8) $G(x_n, x_m, x_m) \le c$ for $n \ge m$. Hence $\{x_n\}$ is a Cauchy sequence. But X is complete. Therefore there exist an $z \in X$ such that $x_n \to z$.

Now we shall prove $Tz = z$, we have by (5)

$$G(x_n, Tz, Tz) \le k\,G(x_{n-1}, Tz, Tz)$$

Letting $n \to \infty$ we get

$$G(z, Tz, Tz) \leq k\, G(z, Tz, Tz)$$

That is

$$(1-k)\, G(z, Tz, Tz) \leq 0$$

But (1-k) > 0, therefore G (z, Tz, Tz) = 0. Hence Tz = z. Therefore z is a fixed point of T. To prove z is unique. If possible z' is another fixed point of T, therefore $Tz' = z'$. Now

$$G(z, z', z') = G(Tz, Tz', Tz') \leq k\, G(z, Tz', Tz') = kG(z, z', z')$$

That is

$$(1-k)\, G(z, z', z') \leq 0$$

But 1 – k > 0, hence $G(z, z', z') = 0$ which implies $z = z'$.

By (6) we have

$$G(Tx, Ty, Ty) \leq k\, G(y, Tx, Tx)$$

Hence

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= k\, G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq k\, G(x_n, Tx_{n-1}, Tx_{n-1}) \\ &= k\, G(x_n, x_n, x_n) \\ &= 0 \end{aligned}$$

Therefore $x_n = x_{n+1}$ for each n. Therefore $\{x_n\}$ converges to x_0 and is a unique fixed point of T.

For case (3) we have

$$G(Tx, Ty, Ty) \leq k\, G(y, Ty, Ty)$$

Hence

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1})$$

That is

$$(1-k)\, G(x_n, x_{n+1}, x_{n+1}) \leq 0$$

Since 1-k > 0, G (x, x, x) = 0 which implies $x_n = x_{n+1}$ for each n. Hence $\{x_n\}$ converges to x_0 and x_0 is a fixed point of T. Hence the theorem.

Corollary: 3.1 Let (X, G) be a complete G – cone metric space and let T: X $\to$ X be a mapping satisfying

$$G(Tx, Ty, Tz) \leq k \vee \{G(x, Ty, Ty),\ G(x, Tz, Tz),\ G(y, Tx, Tx)\ G(y, Tz, Tz), G(z, Tx, Tx),\ G(z, Ty, Ty)\} \quad (9)$$

For all x, y, z $\in$ X, where k $\in$ [0, 1). Then T has a unique fixed point.

Proof: Put z = y in (9) we get

$$G(Tx, Ty, Ty) \leq k \vee \{G(x, Ty, Ty), G(y, Tx, Tx), G(y, Ty, Ty)\} \quad (10)$$

Hence by the theorem (2) T has a unique fixed point.

Corollary: 3.2 Let (X, G) be a complete G – cone metric space and let T : X $\to$ X be a mapping satisfying

$$\begin{aligned} G(T^m x, T^m y, T^m z) \leq k \vee \{&G(x, T^m y, T^m y), G(x, Tz, T^m z), \\ &G(y, T^m x, Tx), G(y, T^m z, T^m z), \\ &G(z, T^m x, T^m x), G(z, T^m y, T^m y) \end{aligned} \quad (11)$$

For all x, y, z $\in$ X, for some m $\in$ N, k $\in$ [0, 1). Then T has a unique fixed point.

Proof: Let y = z, then (11) becomes

$$G(T^m x, T^m y, T^m y) \leq k \vee \{G(x, T^m y, T^m y), G(y, T^m x, T^m x), G(y, T^m y, T^m y)\} \quad (12)$$

By theorem (2) T^m has a unique fixed point z. Again $T^m(Tz) = T^{m+1}z = T(T^m z)$. Therefore Tz is also a fixed point of T. But fixed point is unique. Therefore Tz = z. Hence T has a unique fixed point.

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