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THE PERTURBATION BOUNDS OF THE QR DECOMPOSITION OF MATRICES UNDER THE ADDITION PERTURBATION

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ABSTRACT

The perturbation bounds of the polar decomposition of matrices have been discussed by lots of scholars, such as the article $[1],[2],[3]$. *In this paper the methods in article* $[1]$ *will be extended, according to these methods the QR decomposition of matrices will be discussed when given an addition perturbation to them. In the QR decomposition of matrices the perturbation bounds* $\left\|\overline{R}-R\right\|_2$ *will be given, which can realize on account of the properties of norm and the expression of* R, $\overline{R}$; *the bounds* $\left\|\overline{Q}-Q\right\|_2$ *will also be deduced by using the singular value decomposition of matrices.*

Keywords: *The QR decomposition; The singular value decomposition; Spectral norm; The perturbation bounds.*

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1. INTRODUCTION:

In article [1], the author introduces the polar decomposition: Let $A\in C^{m\times n}$, if there have a unitary matrix $Q\in C^{m\times n}$ and a semi-positive matrix $H\in C^{n\times n}$, such that $A=QH$, we call it the polar decomposition of A. The perturbation bounds of the factors under the addition perturbation have already been given in the form of equality, they are the expression of the singular value decomposition and the perturbation matrix E of A.

This paper only discuss the addition perturbation, namely $\overline{A}=A+E$, which has the same rank of A. We will introduce the QR decomposition of matrices, when transfer the methods in article $[1]$ to the QR decomposition of matrices, we will deduce the perturbation bounds of the unitary factor and the triangle factor of this decomposition in the form of inequality , as a bound inequality is better than equality. These bounds do not need us to know the matrix E, just to know the singular value decompositions of A and $\overline{A}$, the expressions of R and $\overline{R}$. The expressions of R and $\overline{R}$ can come true with the help of the method of $Schmidt$ orthogonalization. All of these can realize by using the tool of $MATLAB$.

That is to say the goal of this paper is to obtain the perturbation bounds of the QR decomposition of matrices in the form of inequality when give an addition perturbation to matrices.

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In this paper we note I as an identity matrix of some dimension, $C_r^{m\times n}$ $(m \ge n)$ as the set of all $m\times n$ matrices with the rank of r; $\lambda(A)=\{\lambda_i\}_{i=1,\dots,n}$, ($\lambda(\bar{A})=\{\bar{\lambda}_i\}_{i=1,\dots,n}$)as the set of all singular values of A ($\bar{A}$); A^H as the conjugate transpose of A;

$$A = U\begin{pmatrix}\Sigma_1 & 0\\ 0 & 0\end{pmatrix}V^H \quad (1)$$

$$\bar{A} = \bar{U}\begin{pmatrix}\bar{\Sigma}_1 & 0\\ 0 & 0\end{pmatrix}\bar{V}^H \quad (2)$$

as the singular value decomposition of matrix $A \in C_r^{m\times n}$ and $\bar{A} \in C_r^{m\times n}$,

respectively, where $U = (U_1 \quad U_2) \in C^{m\times m}$,

$\bar{U} = (\bar{U}_1 \quad \bar{U}_2) \in C^{m\times m}$, $V = (V_1 \quad V_2) \in C^{n\times n}$

$\bar{V} = (\bar{V}_1 \quad \bar{V}_2) \in C^{n\times n}$ are all unitary matrices, $U_1 \in C_r^{m\times r}$, $\bar{U}_1 \in C_r^{m\times r}$, $V_1 \in C_r^{n\times r}$,

$\bar{V}_1 \in C_r^{n\times r}$, $\Sigma_1 = diag(\lambda_1,\dots,\lambda_r) \in C^{r\times r}$,

$\bar{\Sigma}_1 = diag(\bar{\lambda}_1,\dots,\bar{\lambda}_r) \in C^{r\times r}$ and satisfy

$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r > 0$, $\bar{\lambda}_1 \ge \bar{\lambda}_2 \ge \dots \ge \bar{\lambda}_r > 0$; $\| A \|_2$ as the spectral norm of A, which is a unitary invariant norm, and $\| A \|_2 = \max_{i=1,\dots,n}\{\lambda_i\} = \lambda_1$.

2. CONCEPTS:

Definition: 1 [1][4]. Let $Q \in C^{m\times n}$, if $Q^H Q = I$, then we call Q is a unitary matrix.

Definition: 2 [5][6].Let $A \in C_n^{m\times n}$, if there is a $m\times n$ unitary matrix Q and a $n\times n$ up triangle matrix R with all the diagonal elements are positive numbers, such that

$$A = QR \quad (3)$$

then we say (3) is the QR decomposition of A (sometimes call it the unitary-triangle decomposition of matrix), and this kind of matrix has a unique QR decomposition.

We call Q is the unitary factor of A, R is the triangle factor of A.

For matrix $\bar{A}$, we note the QR decomposition as $\bar{A} = \bar{Q}\bar{R}$.

Remark: For matrix $A \in C_n^{n\times n}$, we have the same conclusion. At this time we can obtain the formulas of Q, $\bar{Q}$, R, $\bar{R}$ by using the method of *Schmidt* orthogonalization.

We write $A=(a_1,a_2,\ldots,a_n)$,

$\bar{A}=(\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n)$ First, where a_i and $\bar{a}_i$ are the ith column vector of A and $\bar{A}$ respectively $(i=1,2,\ldots,n)$, then

$$Q=(q_1,q_2,\ldots,q_n) \tag{4}$$

$$\bar{Q}=(\bar{q}_1,\bar{q}_2,\ldots,\bar{q}_n) \tag{4'}$$

$$R=(\lambda_{ji}\,\|p_i\|_2)_{n\times n}=R_1R_2$$

$$=diag(\|p_1\|_2,\|p_2\|_2,\ldots,\|p_n\|_2)\cdot\begin{pmatrix}1 & \lambda_{21} & \lambda_{31} & \ldots & \lambda_{n1}\\ 0 & 1 & \lambda_{32} & \ldots & \lambda_{n2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1\end{pmatrix} \tag{5}$$

$$\bar{R}=(\bar{\lambda}_{ji}\,\|\bar{p}_i\|_2)_{n\times n}=\bar{R}_1\bar{R}_2$$

$$=diag(\|\bar{p}_1\|_2,\|\bar{p}_2\|_2,\ldots,\|\bar{p}_n\|_2)\cdot\begin{pmatrix}1 & \bar{\lambda}_{21} & \bar{\lambda}_{31} & \ldots & \bar{\lambda}_{n1}\\ 0 & 1 & \bar{\lambda}_{32} & \ldots & \bar{\lambda}_{n2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1\end{pmatrix} \tag{5'}$$

where $p_i=a_i-\sum_{j=1}^{i-1}\lambda_{ij}p_j$, $\bar{p}_i=\bar{a}_i-\sum_{j=1}^{i-1}\bar{\lambda}_{ij}\bar{p}_j$, $\|p_i\|_2\neq 0$, $\|\bar{p}_i\|_2\neq 0$, $\lambda_{ij}=\dfrac{(a_i,p_j)}{(p_j,p_j)}$,

$$\bar{\lambda}_{ij}=\frac{(\bar{a}_i,\bar{p}_j)}{(\bar{p}_j,\bar{p}_j)}\quad (i>j)\,,\, q_i=\frac{p_i}{\|p_i\|_2},\quad \bar{q}_i=\frac{\bar{p}_i}{\|\bar{p}_i\|_2}\quad (i=1,2,\ldots,n).$$

Corollary:1 The conditions are the same as definition 2, if matrix Q becomes a diagonal matrix, then the QR decomposition becomes a polar decomposition.

According to corollary 1 and some theorems in article[1], we can obtain the perturbation bounds of these special matrices. Now we will discuss the perturbation bounds of the general QR decomposition of matrices.

3. THE PERTURBATION BOUNDS OF MATRICES UNDER ADDITION PERTURBATION:

3.1. The perturbation bounds of the triangle factor:

Theorem: 1 Let $A\in C_n^{m\times n}$,

$\bar{A}=A+E\in C_n^{m\times n}$, where E is the perturbation matrix of A, $A=QR$,

$\bar{A}=\bar{Q}\bar{R}$, the singular value decomposition of A and $\bar{A}$ are the formulas (1) and (2), then

$$\|\bar{R}-R\|_2\leq \max_{i=1,\ldots,n}\{\big|\|\bar{p}_i\|_2-\|p_i\|_2\big|\}\|\bar{R}_2\|_2+\max_{i=1,\ldots,n}\{\|p_i\|_2\}\|\bar{R}_2-R_2\|_2 \tag{6}$$

$$\|\bar{R}-R\|_2\leq \max_{i=1,\ldots,n}\{\big|\|\bar{p}_i\|_2-\|p_i\|_2\big|\}\|R_2\|_2+\max_{i=1,\ldots,n}\{\|\bar{p}_i\|_2\}\|\bar{R}_2-R_2\|_2 \tag{6'}$$

Proof: For the formula (5) and $(5')$, we have $A = QR = QR_1R_2$ and

$\bar{A} = \bar{Q}\bar{R} = \bar{Q}\bar{R}_1\bar{R}_2$, then

$$\begin{aligned}\|\bar{R} - R\|_2 &= \|\bar{R}_1\bar{R}_2 - R_1R_2\|_2 \\ &= \|\bar{R}_1\bar{R}_2 - R_1\bar{R}_2 + R_1\bar{R}_2 - R_1R_2\|_2 \\ &\le \|\bar{R}_1 - R_1\|_2\|\bar{R}_2\|_2 + \|R_1\|_2\|\bar{R}_2 - R_2\|_2 \end{aligned} \quad (7)$$

$$\begin{aligned}\|\bar{R} - R\|_2 &= \|\bar{R}_1\bar{R}_2 - R_1R_2\|_2 \\ &= \|\bar{R}_1\bar{R}_2 - \bar{R}_1R_2 + \bar{R}_1R_2 - R_1R_2\|_2 \\ &\le \|\bar{R}_2 - R_2\|_2\|\bar{R}_1\|_2 + \|R_2\|_2\|\bar{R}_1 - R_1\|_2 \end{aligned} \quad (7')$$

while R_1 and $\bar{R}_1$ are both diagonal matrix, the singular values of R_1 and $\bar{R}_1$ are

$\lambda(R_1) = \{\|p_i\|_2\}$, $\lambda(\bar{R}_1) = \{\|\bar{p}_i\|_2\}$, $(i = 1, 2, \ldots, n)$, then

$$\|R_1\|_2 = \max_{i=1,\ldots,n}\{\|p_i\|_2\}$$

$$\|\bar{R}_1\|_2 = \max_{i=1,\ldots,n}\{\|\bar{p}_i\|_2\}$$

$$\|\bar{R}_1 - R_1\|_2 = \max_{i=1,\ldots,n}\{|\|\bar{p}_i\|_2 - \|p_i\|_2|\}$$

combine these with (7) and $(7')$, we get the conclusion (6) and $(6')$.

3.2 The perturbation bounds of the unitary factor:

Theorem: 2 The conditions are the same as theorem 1, for the unitary factor we have

$$\|\bar{Q} - Q\|_2 \le \left\|\begin{pmatrix} \bar{\Sigma}_1\bar{V}^HV - \bar{U}_1^HU_1\Sigma_1 \\ -\bar{U}_2^HU_1\Sigma_1 \end{pmatrix}\right\|_2\|\bar{R}^{-1}\|_2 + \max_{i=1,\ldots,n}\{\lambda_i\}\|\bar{R}^{-1} - R^{-1}\|_2 \quad (8)$$

$$\|\bar{Q} - Q\|_2 \le \left\|\begin{pmatrix} U_1^H\bar{U}_1\bar{\Sigma}_1 - \Sigma_1V^H\bar{V} \\ U_2^H\bar{U}_1\bar{\Sigma}_1 \end{pmatrix}\right\|_2\|\bar{R}^{-1}\|_2 + \max_{i=1,\ldots,n}\{\lambda_i\}\|\bar{R}^{-1} - R^{-1}\|_2 \quad (8')$$

Proof: We know that the QR decomposition of matrix A and $\bar{A}$ are the formulas (3) and $(3')$, the singular value decomposition of them are (1) and (2), then $A = QR = U_1\Sigma_1V^H$,

$\bar{A} = \bar{Q}\bar{R} = \bar{U}_1\bar{\Sigma}_1\bar{V}^H$, both R and $\bar{R}$ are up-triangle matrices with positive diagonal elements, that is to say that they are both reversible matrices.

$$\|\bar{Q} - Q\|_2 = \|\bar{U}_1\bar{\Sigma}_1\bar{V}^H\bar{R}^{-1} - U_1\Sigma_1V^HR^{-1}\|_2$$

$$=\|\bar{U}_1\bar{\Sigma}_1\bar{V}^H\bar{R}^{-1}-U_1\Sigma_1V^H\bar{R}^{-1}+U_1\Sigma_1V^H\bar{R}^{-1}-U_1\Sigma_1V^HR^{-1}\|_2$$

$$\le\|\bar{U}_1\bar{\Sigma}_1\bar{V}^H-U_1\Sigma_1V^H\|_2\|\bar{R}^{-1}\|_2+\|U_1\Sigma_1V^H\|_2\|\bar{R}^{-1}-R^{-1}\|_2 \tag{9}$$

$$\|\bar{U}_1\bar{\Sigma}_1\bar{V}^H-U_1\Sigma_1V^H\|_2=\|\bar{U}^H(\bar{U}_1\bar{\Sigma}_1\bar{V}^H-U_1\Sigma_1V^H)V\|_2$$

$$=\left\|\begin{pmatrix}\bar{U}_1^H\\ \bar{U}_2^H\end{pmatrix}(\bar{U}_1\bar{\Sigma}_1\bar{V}^H-U_1\Sigma_1V^H)V\right\|_2$$

$$=\left\|\begin{pmatrix}\bar{\Sigma}_1\bar{V}^H-\bar{U}_1^HU_1\Sigma_1V^H\\ -\bar{U}_2^HU_1\Sigma_1V^H\end{pmatrix}V\right\|_2$$

$$=\left\|\begin{pmatrix}\bar{\Sigma}_1\bar{V}_1^HV-\bar{U}_1^HU_1\Sigma_1\\ -\bar{U}_2^HU_1\Sigma_1\end{pmatrix}\right\|_2 \tag{10}$$

$$\|U_1\Sigma_1V^H\|_2\le\|U_1\|_2\|\Sigma_1\|_2\|V^H\|_2$$

$$=\max_{i=1,\ldots,n}\{\lambda_i\}=\lambda_1 \tag{11}$$

for the formula $(9)-(11)$, we get the conclusion (8). We can obtain formula $(8')$ in the same way. To obtain this we just replace $\bar{U}^H$ by U^H and replace V by $\bar{V}$.

Remark: By formula (9), we also have

$$\|\bar{Q}-Q\|_2=\|U_1\bar{\Sigma}_1\bar{V}^H\bar{R}^{-1}-U_1\Sigma_1V^HR^{-1}\|_2$$

$$\le\|\bar{U}_1\bar{\Sigma}_1\bar{V}^H\|_2\|\bar{R}^{-1}-R^{-1}\|_2+\|\bar{U}_1\bar{\Sigma}_1\bar{V}^H-U_1\Sigma_1V^H\|_2\|R^{-1}\|_2 \tag{9'}$$

$$\|\bar{Q}-Q\|_2\le\left\|\begin{pmatrix}\bar{\Sigma}_1\bar{V}^HV-\bar{U}_1^HU_1\Sigma_1\\ -\bar{U}_2^HU_1\Sigma_1\end{pmatrix}\right\|_2\|R^{-1}\|_2+\max_{i=1,\ldots,n}\{\bar{\lambda}_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{12}$$

$$\|\bar{Q}-Q\|_2\le\left\|\begin{pmatrix}U_1^H\bar{U}_1\bar{\Sigma}_1-\Sigma_1V^H\bar{V}\\ U_2^H\bar{U}_1\bar{\Sigma}_1\end{pmatrix}\right\|_2\|R^{-1}\|_2+\max_{i=1,\ldots,n}\{\bar{\lambda}_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{12'}$$

Corollary: 1 Let $A\in C_n^{n\times n}$ and $\bar{A}=A+E\in C_n^{n\times n}$, then (8) and $(8')$ becomes

$$\|\bar{Q}-Q\|_2\le\|\bar{\Sigma}\bar{V}^HV-\bar{U}^HU\Sigma\|_2\|\bar{R}^{-1}\|_2+\max_{i=1,\ldots,n}\{\lambda_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{13}$$

$$\|\bar{Q}-Q\|_2\le\left\|U^H\bar{U}\bar{\Sigma}-\Sigma V^H\bar{V}\right\|_2\|\bar{R}^{-1}\|_2+\max_{i=1,\ldots,n}\{\lambda_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{13'}$$

respectively. (12) and $(12')$ becomes

$$\|\bar{Q}-Q\|_2\le\|\bar{\Sigma}\bar{V}^HV-\bar{U}^HU\Sigma\|_2\|R^{-1}\|_2+\max_{i=1,\ldots,n}\{\bar{\lambda}_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{14}$$

$$\|\bar{Q}-Q\|_2\le\left\|U^H\bar{U}\bar{\Sigma}-\Sigma V^H\bar{V}\right\|_2\|R^{-1}\|_2+\max_{i=1,\ldots,n}\{\bar{\lambda}_i\}\|\bar{R}^{-1}-R^{-1}\|_2 \tag{14'}$$

respectively.

Corollary: 2 The conditions are the same of theorem 2, then for the formula (8) and (12), we have

$$\left\|\|\bar{R}^{-1}\|_2 - \|R^{-1}\|_2\right\| \|\bar{Q}-Q\|_2 \le \left|\max_{i=1,\dots,n}\{\bar{\lambda}_i\} - \max_{i=1,\dots,n}\{\lambda_i\}\right|$$

$$\|\bar{R}^{-1} - R^{-1}\|_2 \left\|\|\bar{R}^{-1}\|_2 - \|R^{-1}\|_2\right| \tag{15}$$

Proof: For formula (8), we have

$$\|R^{-1}\|_2 \, \|\bar{Q}-Q\|_2 \le \left\|\begin{pmatrix}\bar{\Sigma}_1\bar{V}^H V - \bar{U}_1^H U_1\Sigma_1 \\ -\bar{U}_2^H U_1\Sigma_1\end{pmatrix}\right\|_2 \|\bar{R}^{-1}\|_2 \, \|R^{-1}\|_2 + \max_{i=1,\dots,n}\{\lambda_i\} \|\bar{R}^{-1} - R^{-1}\|_2 \, \|R^{-1}\|_2 \tag{16}$$

For formula (12), we have

$$\|\bar{R}^{-1}\|_2 \, \|\bar{Q}-Q\|_2 \le \left\|\begin{pmatrix}\bar{\Sigma}_1\bar{V}^H V - \bar{U}_1^H U_1\Sigma_1 \\ -\bar{U}_2^H U_1\Sigma_1\end{pmatrix}\right\|_2 \|R^{-1}\|_2 \, \|\bar{R}^{-1}\|_2 + \max_{i=1,\dots,n}\{\bar{\lambda}_i\} \|\bar{R}^{-1} - R^{-1}\|_2 \, \|\bar{R}^{-1}\|_2 \tag{16'}$$

$(16')$ subtracts (16), we can obtain the conclusion.

Remark: If $\|\bar{R}^{-1}\|_2 - \|R^{-1}\|_2 \ne 0$, then (15) becomes

$$\|\bar{Q}-Q\|_2 \le \left|\max_{i=1,\dots,n}\{\bar{\lambda}_i\} - \max_{i=1,\dots,n}\{\lambda_i\}\right| \cdot \|\bar{R}^{-1} - R^{-1}\|_2 \tag{15'}$$

Replace $\|\bar{R}^{-1} - R^{-1}\|_2$ by formula (6)

or $(6')$, namely the perturbation bounds

of R, we can obtain another result.

4. CONCLUSION:

The main point of this paper is the perturbation bounds of the unitary and triangle factor of QR decomposition of matrices under the addition perturbation, though the upper-bounds of both are not the smallest, they can be a standard for measuring the perturbation of matrices. They are useful in some aspects for calculating the algebraic equations.

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