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The Pell equation $x^2 - Dy^2 = \pm 9$

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ABSTRACT

Let $D \neq 1$ be a positive non-square integer. Extending the work of A. Tekcan, here we obtain some formulas for the integer solutions of the Pell equation $x^2 - Dy^2 = \pm 9$.

Keywords: *Pell's equation, solutions of Pell's equation.*

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1. INTRODUCTION:

The equation $x^2 - Dy^2 = N$, with given integers D and N and unknowns x and y, is called Pell's equation. If D is negative, it can have only a finite number of solutions. If D is a perfect square, say $D = a^2$, the equation reduces to $(x - ay)(x + ay) = N$ and again there is only a finite number of solutions. The most interesting case of the equation arises when $D \neq 1$ be a positive non-square.

Although J. Pell contributed very little to the analysis of the equation, it bears his name because of a mistake by Euler. Pell's equation $x^2 - Dy^2 = 1$ was solved by Lagrange in terms of simple continued fractions. Lagrange was the first to prove that $x^2 - Dy^2 = 1$ has infinitly many solutions in integers if $D \neq 1$ is a fixed positive non-square integer. If the lenght of the periode of $\sqrt{D}$ is l, all positive solutions are given by $x = P_{2\mathrm{vk}-1}$ and $y = Q_{2\mathrm{vk}-1}$ if k is odd, and by $x = P_{\mathrm{vk}-1}$ and $y = Q_{\mathrm{vk}-1}$ if k is even, where $\mathrm{v} = 1, 2, \ldots$ and $\frac{P_n}{Q_n}$ denotes the nth convergent of the continued fraction expansion of $\sqrt{D}$. Incidentally, $x = P_{(2\mathrm{v}-1)(\mathrm{k}-1)}$ and $y = Q_{(2\mathrm{v}-1)(\mathrm{k}-1)}$, $\mathrm{v} = 1, 2, \ldots$ are the positive solutions of $x^2 - Dy^2 = -1$ provided that v is odd.

There is no solution of $x^2 - Dy^2 = \pm 1$ other than $x_\mathrm{v}, y_\mathrm{v}$: $\mathrm{v} = 1, 2, \ldots$ given by $(x_1 + \sqrt{D}\, y_1)^\mathrm{v} = x_\mathrm{v} + \sqrt{D}\, y_\mathrm{v}$, where (x_1, y_1) is the least positive solution called the fundamental solution, which there are different method for finding it. The reader can find many references in the subject in the book [I].

For completeness we recall that there are many papers in which are considered different types of Pell's equation. Many authors such as Tekcan [A], Kaplan and Williams [K], Matthews [M], Mollin, Poorten and Williams [P], Stevenhagen [S] and the others consider eome specific Pell equations and their integer solutions. A. Tekcan in [A], considered the equation $x^2 - Dy^2 = \pm 4$ and he obtained some formulas for its integer solutions. He mentioned two conjecture which was proved by A. S. Shabani [Ss].

In this paper we extend the work of A. Tekcan by considering the Pell equation $x^2 - Dy^2 = \pm k^2$ when Let $D \neq 1$ be a positive non-square integer and $k \geq 2$, we obtain some formulas for its integer solutions.

2. The Pell equation $x^2 - Dy^2 = 9$:

In this section, we consider the solutions of Pell's equation $x^2 - Dy^2 = 9$.

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Theorem: 2.1 *Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = 9$, and let*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

for $n \geq 1$. Then the integer solutions of the Pell equation $x^2 - Dy^2 = 9$ are (x_n, y_n), where

$$(x_n, y_n) = \left(\frac{u_n}{3^{n-1}}, \frac{v_n}{3^{n-1}}\right) \quad (2)$$

Proof: We prove the theorem using the method of mathematical induction. For $n = 1$, we have from (1), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = 9$. Now, we assume that the Pell equation $x^2 - Dy^2 = 9$ is satisfied for (x_{n-1}, y_{n-1}), i.e.

$$x_{n-1}{}^2 - Dy_{n-1}{}^2 = \frac{u_{n-1}{}^2 - Dv_{n-1}{}^2}{3^{2n-4}} = 9 \quad (3)$$

and we show that it holds for (x_n, y_n). Indeed, by (1), it is easy to prove that

$$\begin{cases} u_n = x_1 u_{n-1} + D\, y_1 v_{n-1} \\ v_n = y_1 u_{n-1} + x_1 v_{n-1} \end{cases} \quad (4)$$

Hence,

$$x_n{}^2 - Dy_n{}^2 = \frac{u_n{}^2 - Dv_n{}^2}{3^{2n-2}} = (x_1{}^2 - Dy_1{}^2)\frac{(u_{n-1}{}^2 - Dv_{n-1}{}^2)}{3^{2n-2}}$$

Applying (3) it is easily seen that $u_{n-1}{}^2 - Dv_{n-1}{}^2 = 3^{2n-2}$. Hence we conclude that $x_n{}^2 - Dy_n{}^2 = x_1{}^2 - Dy_1{}^2 = 9$. Therefore (x_n, y_n) is also a solution of the Pell equation $x^2 - Dy^2 = 9$. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = 9$.

Corollary: 2.2 *Let (x_1, y_1) is the fundamental solution of the Pell equation $x^2 - Dy^2 = 9$, then*

$$x_n = \frac{x_1 x_{n-1} + Dy_1 y_{n-1}}{3}, \quad y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{3}, \quad (5)$$

and

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -3y_1 \quad (6)$$

Proof: By (1), we have $u_n = x_1 u_{n-1} + Dy_1 v_{n-1}$ and $v_n = y_1 u_{n-1} + x_1 v_{n-1}$ by (2), we have $u_n = 3^{n-1} x_n$ and $v_n = 3^{n-1} y_n$. We get

$$u_n = x_1 u_{n-1} + Dy_1 v_{n-1},$$

then,

$$3^{n-1} x_n = x_1 3^{n-2} x_{n-2} + Dy_1 3^{n-2} y_{n-1},$$

witch gives

$$x_n = \frac{x_1 x_{n-1} + Dy_1 y_{n-1}}{3}.$$

Similarly, we have

$$y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{3}.$$

and hence

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = x_n y_{n-1} - y_n x_{n-1} = -\frac{y_1(x_{n-1}{}^2 - Dy_{n-1}{}^2)}{3} = -3y_1$$

Theorem: 2.3 *Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = 9$, then (x_n, y_n) satisfy the following recurrence relations*

$$\begin{cases} x_n = \left(\frac{2}{3}x_1 - 1\right)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n = \left(\frac{2}{3}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3} \end{cases} \quad (7)$$

for $n \geq 4$.

Proof. The proof will be by induction on n. Using (5), we have

$$\begin{cases} x_2 = \frac{2}{3}{x_1}^2 - 3\,, \\ y_2 = \frac{2}{3}x_1 y_1. \end{cases} \quad (8)$$

Using (5) and (8), we get

$$\begin{cases} x_3 = x_1\left(\frac{4}{9}{x_1}^2 - 3\right) \\ y_3 = y_1\left(\frac{4}{9}{x_1}^2 - 3\right) \end{cases} \quad (9)$$

Then by (5) and (9), we find x_4 and y_4.

$$\begin{cases} x_4 = \frac{8}{27}{x_1}^4 - \frac{8}{3}{x_1}^2 + 3 \\ y_4 = y_1 x_1\left(\frac{8}{27}{x_1}^2 - \frac{4}{3}\right) \end{cases} \quad (10)$$

Now, replacing (8) and (9), in (7), one obtains

$$x_4 = \left(\frac{2}{3}x_1 - 1\right)(x_3 + x_2) - x_1 = \frac{8}{27}{x_1}^4 - \frac{8}{3}{x_1}^2 + 3$$

and

$$y_4 = \left(\frac{2}{3}x_1 - 1\right)(y_3 + y_2) - y_1 = y_1 x_1\left(\frac{8}{27}{x_1}^2 - \frac{4}{3}\right)$$

which are the same formulas as in (10). Therefore (7) holds for $n = 4$.

Now, we assume that (7) holds for $n \geq 4$ and we show that it holds for $n + 1$. Indeed, by (5) and by hypothesis we have

$$x_{n+1} = \frac{x_1 x_n + D y_1 y_n}{3} = \left(\frac{2}{3}x_1 - 1\right)(x_n + x_{n-1}) - x_{n-2},$$

and

$$y_{n+1} = \frac{x_1 x_n + x_1 y_n}{3} = \left(\frac{2}{3}x_1 - 1\right)(y_{n-1} + y_{n-2}) - y_{n-3},$$

completing the proof.

3. The negative Pell equation $x^2 - Dy^2 = -9$:

Theorem: 3.1 *Let (x_1, y_1) be the fundamental solution of the Pell equation $x^2 - Dy^2 = -9$, then the other solutions are (x_{2n+1}, y_{2n+1}), where*

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{9^n}, \frac{v_{2n+1}}{9^n}\right) \quad (11)$$

for $n \geq 0$.

We prove the theorem using the method of mathematical induction. For $n = 0$, we have from (11), $(u_1, v_1) = (x_1, y_1)$ which is the fundamental solution of $x^2 - Dy^2 = -9$. Now, we assume that the Pell equation $x^2 - Dy^2 = -9$ is satisfied for $n \geq 0$. So, (x_{2n+1}, y_{2n+1}), i.e.

$${x_{2n+1}}^2 - D\,{y_{2n+1}}^2 = \frac{{u_{2n+1}}^2 - D\,{v_{2n+1}}^2}{9^{2n}} = -9, \quad (12)$$

and we show that it holds for $n + 1$. Indeed, by (1), it is easily to seen that

$$\begin{pmatrix} u_{2n+3} \\ v_{2n+3} \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} ({x_1}^2 + D\,{y_1}^2)u_{2n+1} + 2Dx_1y_1u_{2n+1} \\ 2x_1y_1v_{2n+1} + ({x_1}^2 + D\,{y_1}^2)v_{2n+1} \end{pmatrix}.$$

Hence,

$$(x_{2n+3})^2 - D(\,y_{2n+3})^2 = \frac{(u_{2n+3})^2 - D\,(v_{2n+3})^2}{9^{2n+2}} = -9$$

Therefore $(x_{2(n+1)+1},\ y_{2(n+1)+1}) = (x_{2n+3},\ y_{2n+3})$ is also a solution of the Pell equation $x^2 - Dy^2 = -9$. Since n is arbitrary, we get all integer solutions of the Pell equation $x^2 - Dy^2 = -9$.

Corollary: 3.2 *Let $(x_1,\ y_1)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = -9$, then*

$$x_{2n+1} = \frac{({x_1}^2 + D\,{y_1}^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{9},$$
$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + ({x_1}^2 + D\,{y_1}^2)y_{2n-1}}{9}, \tag{14}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = 2x_1y_1 \tag{15}$$

Proof: Using (1), we have $u_{2n+1} = ({x_1}^2 + D\,{y_1}^2)u_{2n-1} + 2Dx_1y_1v_{2n-1}$ and $v_{2n+1} = 2x_1y_1u_{2n-1} + ({x_1}^2 +$ *D y12v2n−1.* By *11,* we have *u2n+1=9nx2n+1* and *v2n+1=9ny2n+1.* We get

$$u_{2n+1} = ({x_1}^2 + D\,{y_1}^2)u_{2n-1} + 2Dx_1y_1v_{2n-1}$$

then,

$$9^n x_{2n+1} = ({x_1}^2 + D\,{y_1}^2)9^{n-1}x_{2n-1} + 2Dx_1y_1 9^{n-1}y_{2n-1}$$

witch gives

$$x_{2n+1} = \frac{({x_1}^2 + D\,{y_1}^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{9}.$$

Similarly, we have

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + ({x_1}^2 + D\,{y_1}^2)y_{2n-1}}{9}.$$

and hence

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = x_{2n+1}y_{2n-1} - y_{2n+1}x_{2n-1}$$
$$= \frac{({x_1}^2 + D\,{y_1}^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{9}y_{2n-1}$$
$$- \frac{2x_1y_1x_{2n-1} + ({x_1}^2 + D\,{y_1}^2)y_{2n-1}}{9}x_{2n-1}$$
$$= 2x_1y_1\frac{D(\,y_{2n-1})^2 - (\,x_{2n-1})^2}{9} = 2x_1y_1\frac{-(-k^2)}{9}$$
$$= 2x_1y_1$$

Theorem: 2.3 *Let $(x_1,\ y_1)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = -9$, then $(x_n,\ y_n)$ satisfy the following recurrence relations*

$$\begin{cases} x_{2n+1} = \left(\frac{4}{9}{x_1}^2 + 1\right)(x_{2n-1} + x_{2n-3}) - x_{2n-5} \\ y_{2n+1} = \left(\frac{4}{9}{x_1}^2 + 1\right)(y_{2n-1} + y_{2n-3}) - y_{2n-5} \end{cases} \tag{16}$$

for $n\ \geq 3$.

Proof: The proof will be by induction on n. Using (14), we have

$$\begin{cases} x_3 = x_1\left(\frac{4}{9}{x_1}^2 + 3\right) \\ y_3 = y_1\left(\frac{4}{9}{x_1}^2 + 1\right) \end{cases} \qquad (17)$$

Using (14), (17) and (18), we get

$$\begin{cases} x_5 = \frac{4}{9}x_1\left(\frac{4}{9}{x_1}^4 + 5x_1 + 5\frac{9}{4}\right) \\ y_5 = \frac{4}{9}y_1\left(\frac{4}{9}{x_1}^4 + 3{x_1}^2 + \frac{9}{4}\right) \end{cases} \qquad (18)$$

Then by (18), we find x_7 and y_7. So, we obtained

$$\begin{cases} x_7 = \frac{4}{9}x_1\left(\frac{16}{81}{x_1}^6 + \frac{28}{9}{x_1}^4 + 14{x_1}^2 + 7\frac{9}{4}\right) \\ y_7 = \frac{4}{9}y_1\left(\frac{16}{81}{x_1}^6 + 5\frac{4}{9}{x_1}^4 + 6{x_1}^2 + \frac{9}{4}\right) \end{cases} \qquad (19)$$

Now, replacing (17), (18) and (19) in (16), one obtains

$$x_7 = \left(\frac{4}{9}{x_1}^2 + 1\right)(x_5 + x_3) - x_1$$

and

$$y_7 = \left(\frac{4}{9}{x_1}^2 + 1\right)(y_5 + y_3) - y_1$$

which are the same formulas as in (19). Therefore (16) holds for $n = 3$.

Now, we assume that (16) holds for $n \geq 3$ and we show that it holds for $n + 1$. Indeed, by (14) and by hypothesis we have

$$x_{2n+3} = \frac{({x_1}^2 + D\,{y_1}^2)x_{2n+1} + 2Dx_1y_1y_{2n+1}}{9} = \left(\frac{4}{9}{x_1}^2 + 1\right)(x_{2n+1} + x_{2n-1}) - x_{2n-3}$$

$$y_{2n+3} = \frac{2x_1y_1x_{2n+1} + ({x_1}^2 + D\,{y_1}^2)y_{2n+1}}{9} = \left(\frac{4}{9}{x_1}^2 + 1\right)(y_{2n+1} + y_{2n-1}) - y_{2n-3}$$

completing the proof.

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