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JUDGMENT OF FINITE CODIMENSIONAL IDEALS IN E_n AND CALCULATION OF THEIR COMPLEMENTARY SPACES

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ABSTRACT

The study first judged whether an ideal in E_n is finite codimensional by utilizing some relevant conclusions acquired from several propositions of the necessary and sufficient conditions for &-equivalence. Then, the calculation of a basis set of the complementary space of finite codimensional ideals in E_n was studied using certain algebraic knowledge, and the examples were provided.

Keywords: *Ring of Germs of C^∞ Real Functions E_n ; Finite Codimensional Ideals; Complementary Space*

2010 AMS Mathematics Subject Classification: *58K40.*

1. INTRODUCTION AND PROPAEDEUTICS

1.1 INTRODUCTION

In singularity theory, many important issues are determined by the judgment and calculation of finite codimensional ideals in the ring of germs of functions. For example, regarding problems of finite determination, Mather [1] once noted that if the ideal $M_n J(f)$ is finite codimensional in E_n, where $J(f)=\left[\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right]_{E_n}$ is the Jacobian ideal of f and M_n represents the only maximal ideal in E_n, then $f \in E_n$ is finitely determined. For the judgment and analysis of finite codimensional ideals in the ring of germs, Arnold [2], Broker [3], Golubitsky [4] and Martinet [5] drew conclusions that are applicable to the complex analytic ring of germs θ_n. Based on the conclusions of Arnold, Siersman [6] and Cen [7] analyzed the codimension of ideals under the assumption that the ideal in the ring of germs is finite codimensional. In addition, Cen [8] discussed whether there exist higher-order Morse germs in the ring of C^∞ function germs about multidimensional variables, and scholars have discussed the question of germs from various points of view in references [9] to [11].

It is worth noting that Mather [1] proved the following fundamental theorems for the universal deformation of c^∞ real function germs, that is, the germ f has a P-parameter universal deformation of $F(t,x)$, $t=(t_1,\dots,t_p)\in R^p$, $x=(x_1,\dots,x_n)\in R^n$ if and only if $J(f)+R\{F_1,F_2,\dots,F_p\}=E_n$, $J(f)=\left[\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right]_{E_n}$,
i.e., the P-parameter universal deformation of germ f in E_n can be obtained only if the basis of the complementary space of the Jacobian ideal of f in E_n can be calculated. Thus, calculation of the basis of the complementary space of finite codimensional ideals in E_n is very important.

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However, literature regarding methods for determining the basis of the complementary space of finite codimensional ideals in E_n is rare. Meanwhile, whether an ideal in E_n is finite codimensional must be judged. In this study, solutions to these two typical problems, the judgment of finite codimensional ideals in E_n and the calculation of its complementary space, are proposed based on the available relevant analyses and discussions. Then, a more complete answer is proposed.

1.2 PROPAEDEUTICS

Let M_n be the maximal ideal of E_n and M_n be the maximal ideal of θ_n, where θ_n expresses the ring of complex-analytic germs. Let Q_n be the ring of real-analytic germs and M_{Q_n} be the maximal ideal of Q_n, $Q_n \subset \theta_n$, $Q_n \subset E_n$.

Let M_n^k, $M_{\theta_n}^k$ and $M_{Q_n}^k$ be degree k of M_n, M_{θ_n}, and M_{Q_n}, respectively.

Let J_n^k be the quotient algebra $E_n /_{M_n^{k+1}}$, where J_n^k is canonically isomorphic to the algebra of polynomial germs with degree less than or equal to k. If f is a germ in E_n, its projection into J_n^k can be considered to be its Taylor polynomial of order k at $o \in R^n$. This canonical projection is denoted by

$$J^k : E_n \to J_n^k, f \mapsto j^k f .$$

Let P_n^k be the entirety of the homogeneous polynomial germs of degree k in E_n,

$$P_n^k = M_n^k / M_n^{k+1},\ I_n^k = I / M_n^{k+1} = J^k(I),$$

and let V_n be the entirety of the reversible germs in E_n.

Definition 1: Let I be an ideal in E_n. If there exists a natural number k that makes $M_n^k \subset I$, then I is finite codimensional in E_n, namely, $\dim_R E_n / I < +\infty$.

Lemma 1: [8]. Let $f : (R^n,0) \to (R^n,0)$, $(x_1,...,x_n) \mapsto (f_1(x),..., f_n(x))$ be a C^∞ map-germ, where the ideal $\langle f_1,..., f_n \rangle_{E_n}$ is finite codimensional in E_n. Then, there is a natural number $k \in N$ that makes

$$\langle f_1,..., f_n \rangle_{E_n} = \langle j^k f_1, \cdots, j^k f_n \rangle_{E_n} .$$

This lemma indicates that for any finitely generated and finite codimensional ideal in E_n, its generators can be regarded as polynomial germs.

Without loss of generality, if the finite codimensional ideals in E_n are involved, their generators are always considered as polynomial germs in the following discussion.

Lemma 2: [6]. Let I be the finitely generated ideal in E_n, where $I = \langle f_1, f_2,..., f_m \rangle_{E_n}$. If $h_j \in V_n$ $(j = 1, 2,..., m)$,

$$\langle f_1, f_2,..., f_m \rangle_{E_n} = \langle f_1, f_2,..., h_i f_i,..., f_m \rangle_{E_n} = \langle h_1 f_1,..., h_i f_i,..., h_m f_m \rangle_{E_n} .$$

Lemma 3: [3]. Let ξ be a standard basis element in P_n^r, and let $I = \langle \psi_1,...,\psi_q, \varphi_{q+1},..., \varphi_m \rangle_{E_n}$ be a finite codimensional ideal. Then,

$$\xi \in I \oplus R\{g_1,..., g_s\}$$

if and only if there are the polynomial germs $\eta_1,...,\eta_q$ in E_n for which the following hold:

(i) $H^r(\sum_{i=1}^{q}\eta_i j^{k-1}\psi_i)$ contains the term ξ.

(ii) $\left\{H^r(\sum_{i=1}^{q}\eta_i j^{k-1}\psi_i)-\xi\right\}\in \mathrm{I}$.

If it is difficult to directly determine $\xi\in I$, the following theorem provides a better method.

2. MAIN RESULTS

First, whether an ideal in E_n is finite codimensional is judged in this study according to the above propositions. Then, relevant conclusions acquired from several propositions regarding the necessary and sufficient conditions for $\&$-equivalence are used as a foundation. The calculation of the basis of the complementary space of finite codimensional ideals in E_n are studied using certain algebraic knowledge.

2.1 Judgment of finite codimensional ideals in E_n

Theorem 1: Assume $f_1(x),\cdots,f_n(x)$ to be the polynomial germs in E_n, $x=(x_1,\cdots,x_n)\in R^n$. Then,

$$\dim_R E_n/\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}<\infty$$

if and only if

$$\dim_R Q_n/\langle f_1(x),\cdots,f_n(x)\rangle_{Q_n}<\infty.$$

Proof: Necessity: Because

$$\dim_R E_n/\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}<\infty,$$

there exists $k\in N$ such that

$$M_n^k\subset\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}.$$

All degree-k monomial germs $x_1^{h_1}\cdots x_n^{h_n}$ about $x_1\cdots x_n$ ($\sum_{i=1}^{n}h_i=k$, where h_i are nonnegative integers) compose a team of generators of M_n^k. Thus, there exist $a_i\in E_n$, $i=1,\cdots,n$, for each $x_1^{h_1}\cdots x_n^{h_n}$ such that

$$x_1^{h_1}\cdots x_n^{h_n}=\sum_{i=1}^{n}a_i\cdot f_i(x).$$

Because $x_1^{h_1}\cdots x_n^{h_n}$ and $f_1(x),\cdots,f_n(x)$ are polynomial germs, if each a_i is replaced by its k-th Taylor polynomial germ $j^k a_i$, one can deduce that

$$x_1^{h_1}\cdots x_n^{h_n}=\sum_{i=1}^{n}j^k a_i\cdot f_i(x)\bmod{ulo}M_{Q_n}^{k+1},\ j^k a_i\in Q_n,\quad i=1,\cdots,n$$

and

$$M_{Q_n}^k\subset\langle f_1(x),\cdots,f_n(x)\rangle_{Q_n}+M_{Q_n}^{k+1}.$$

By the Nakayama lemma, one can deduce that

$$M_{Q_n}^k\subset\langle f_1(x),\cdots,f_n(x)\rangle_{Q_n}.$$

Thus,

$$\dim_R Q_n/\langle f_1(x),\cdots,f_n(x)\rangle_{Q_n}<\infty.$$

Sufficiency: There exists $k \in N$ such that

$$M_{Q_n}^k \subset \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n}$$

by virtue of

$$\dim_R Q_n / \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n} < \infty .$$

Because all degree-k monomials $x_1^{h_1} \cdots x_n^{h_n}$ ($\sum_{i=1}^{n} h_i = k$, where h_i are nonnegative integers) compose a team of generators of $M_{Q_n}^k$, there exist $a_i \in Q_n \subset E_n$, $i = 1, \cdots, n$, for each $x_1^{h_1} \cdots x_n^{h_n}$ such that

$$x_1^{h_1} \cdots x_n^{h_n} = \sum_{i=1}^{n} a_i \cdot f_i(x).$$

The monomial germs $x_1^{h_1} \cdots x_n^{h_n}$ ($\sum_{i=1}^{n} h_i = k$, where h_i are nonnegative integers) also compose a team of generators of M_n^k. Hence,

$$M_n^k \subset \langle f_1(x), \cdots, f_n(x) \rangle_{E_n} ,$$

that is,

$$\dim_R Q_n / \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n} < \infty .$$

Theorem 2: Assume that $f_1(x), \cdots, f_n(x)$ are the polynomial germs in E_n, $x = (x_1, \cdots, x_n) \in R^n$, and $f_i(z)$ have the same form as $f_i(x)$, where $z = (z_1, \cdots, z_n) \in c^n$ is replaced from the complex field, $i = 1, \cdots, n.$ Then,

$$\dim_R E_n / \langle f_1(x), \cdots, f_n(x) \rangle_{E_n} < \infty$$

if and only if

$$\dim_c \theta_n / \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n} < \infty .$$

Proof: Necessity: Because $\langle f_1(x), \cdots, f_n(x) \rangle_{E_n}$ has finite codimension in E_n, there exists $k \in N$ such that

$$M_{Q_n}^k \subset \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n}$$

by virtue of theorem 1.

For a team of generators of $M_{Q_n}^k$, there exist $a_i(x) \in Q_n$ for each generator $x_1^{h_1} \cdots x_n^{h_n}$ such that

$$x_1^{h_1} \cdots x_n^{h_n} = \sum_{i=1}^{n} a_i(x) \cdot f_i(x).$$

When the variable $x \in R^n$ in the above equations is replaced by $z = (z_1, \cdots, z_n) \in c^n$ from the complex field, the equation becomes

$$z_1^{h_1} \cdots z_n^{h_n} = \sum_{i=1}^{n} a_i(z) \cdot f_i(z),\ a_i(x) \in Q_n,\ f_i(z) \in Q_n,\ i = 1, \cdots, n .$$

Because all degree-k monomial germs $z_1^{h_1} \cdots z_n^{h_n}$ ($\sum_{i=1}^{n} h_i = k$, where h_i are nonnegative integers) constitute a team of generators of $M_{\theta_n}^k$, one can infer that

$$M_{\theta_n}^k \subset \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n} .$$

Therefore,

$$\dim_c \theta_n / \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n} < \infty .$$

Sufficiency: There exists $k \in N$ such that

$$M_{\theta_n}^k \subset \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n}$$

because

$$\dim_c \theta_n / \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n} < \infty .$$

However, all degree-k monomials $z_1^{h_1} \cdots z_n^{h_n}$ ($\sum_{i=1}^{n} h_i = k$, where h_i are nonnegative integers) about variable z form a team of generators of $M_{\theta_n}^k$.

There exist $a_i(z) \in \theta_n$ for each $z_1^{h_1} \cdots z_n^{h_n}$ such that

$$z_1^{h_1} \cdots z_n^{h_n} = \sum_{i=1}^{n} a_i(z) f_i(z),\ i = 1, \cdots, n .$$

Because $z_1^{h_1} \cdots z_n^{h_n}$ and $f_i(z)$, $i = 1, \cdots, n$ are all real-coefficient germs in θ_n, $a_i(z)$ are also real-coefficient germs in θ_n, $i = 1, \cdots, n$. When $z = (z_1, \cdots, z_n) \in c^n$ is taken to be real in the following equation

$$z_1^{h_1} \cdots z_n^{h_n} = \sum_{i=1}^{n} a_i(z) f_i(z),$$

the equation is transformed to

$$x_1^{h_1} \cdots x_n^{h_n} = \sum_{i=1}^{n} a_i(x) f_i(x),\ a_i(x) \in Q_n,\ i = 1, \cdots, n .$$

Consequently, one can deduce that

$$M_{Q_n}^k \subset \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n}$$

and

$$\dim_R Q_n / \langle f_1(x), \cdots, f_n(x) \rangle_{Q_n} < \infty .$$

Therefore,

$$\dim_R E_n / \langle f_1(x), \cdots, f_n(x) \rangle_{E_n} < \infty$$

by theorem 1.

Theorem 2 establishes the foundation for exploring whether the ideal generated by polynomial germs in E_n is codimensional.

Lemma 4: [4]. Assume that

$$f = (f_1(z), \cdots, f_n(z)) : (c^n, a) \to (c^n, 0)$$

be a holographic mapping germ, where $f_1(z), \cdots, f_n(z)$ are the components of f, $f_i(z) \in \theta_n$, $i = 1, ..., n$. Then,

$$\dim_c \theta_n / \langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n} < \infty$$

if and only if the null point of $\langle f_1(z), \cdots, f_n(z) \rangle_{\theta_n}$ is isolated in c^n, i.e., the solution of the system of equations

$$\begin{cases} f_1(z_1, \cdots, z_n) = 0 \\ \cdots \\ f_n(z_1, \cdots, z_n) = 0 \end{cases}$$

is isolated in c^n.

Theorem 3: Let $f:(R_n,0)\to(R_n,0)$, $(x_1,...,x_n)\mapsto(f_1(x),...,f_n(x))$ be a real polynomial map-germ, where $f_1(x),\cdots f_n(x)$ are polynomial germs in E_n, and $f_i(z)$ have the same form as $f_i(x)$, where $z=(z_1,\cdots,z_n)\in c^n$ is taken from the complex field, $i=1,\cdots,n$. Then,

$$\dim_R E_n/\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}<\infty$$

if and only if the null points of $\langle f_1(z),\cdots,f_n(z)\rangle_{\theta_n}$ are isolated in c^n, i.e., the solution of the system of equations

$$\begin{cases} f_1(z_1,\cdots,z_n)=0\\ \cdots\\ f_n(z_1,\cdots,z_n)=0\end{cases}$$

is isolated in c^n.

Proof: Because $f_i(x)\in Q_n$ and $f_i(x)\in\theta_n$, $i=1,\cdots,n$, one can deduce that

$$\dim_c \theta_n/\langle f_1(z),\cdots,f_n(z)\rangle_{\theta_n}<\infty$$

if and only if the null point of $\langle f_1(z),\cdots,f_n(z)\rangle_{\theta_n}$ is isolated in c^n by virtue of lemma 4.

According to theorem 2, one can deduce that

$$\dim_c \theta_n/\langle f_1(z),\cdots,f_n(z)\rangle_{\theta_n}<\infty$$

if and only if

$$\dim_R E_n/\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}<\infty.$$

Hence,

$$\dim_R E_n/\langle f_1(x),\cdots,f_n(x)\rangle_{E_n}<\infty$$

if and only if the null point of $\langle f_1(z),\cdots,f_n(z)\rangle_{\theta_n}$ is isolated in c^n.

2.2 Calculation for the complementary space on a finite codimensional ideal in E_n

Problem 1: Assume that

$$I=\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2+zy^3+x^4y,\ xyz+xy^3\rangle$$

be an ideal in E_3, judge whether I is codimensional in E_3, if so, calculate a team of base of the complementary space of finite codimensional ideal I in E_3.

One can solve this problem with two different methods.

Firstly, by lemma 2,

$$I\overset{(2)-xy(1)}{=}\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2+zy^3+x^4y-xy(x^3-3x^3yz^2+y^3z^2+z^8),\ xyz+xy^3\rangle$$

$$=\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2+zy^3+3x^4y^2z^2-xy^4z^2-xyz^8,\ xyz+xy^3\rangle$$

$$\overset{(2)+z^7(3)}{=}\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2+zy^3+3x^4y^2z^2-xy^4z^2+xy^3z^7,\ xyz+xy^3\rangle$$

$$=\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2(1+zy+3x^4z^2-xy^2z^2+xyz^7),\ xyz+xy^3\rangle$$

$$=\langle x^3-3x^3yz^2+y^3z^2+z^8,\ y^2,\ xyz\rangle$$

$$=\langle x^3+z^8,\ y^2,\ xyz\rangle$$

It is clear that $x^m\notin I$ for any $m\in N$. Therefore I is not codimensional in E_3.

On the other hand, one knows $I=\left\langle x^3+z^8, y^2, xyz\right\rangle$. According to theorem 3, $0\in C$ is not the isolated null points of

$$\begin{cases} z_1^3+z_3^8=0 \\ \quad z_2^2=0 \qquad , z_1, z_2, z_3 \in c^3 \\ z_1 z_2 z_3=0 \end{cases}$$

and I is not codimensional in E_3.

Problem 2: Assume that

$$I=\left\langle f_1, f_2\right\rangle=\left\langle xy+xy^6+y^8, y^2+x^3-2xy^3\right\rangle$$

be an ideal in E_2, judge whether I is codimensional in E_2, if so, calculate a team of base of the complementary space of finite codimensional ideal I in E_2.

Due to lemma 2,

$$\begin{aligned} I=\left\langle f_1, f_2\right\rangle &=\left\langle f_1-y^6 f_2, f_2\right\rangle=\left\langle xy+xy^6-x^3y^6+2xy^9, y^2+x^3-2xy^3\right\rangle \\ &=\left\langle xy(1+x^5-x^2y^5+2y^8), y^2+x^3-2xy^3\right\rangle \\ &=\left\langle xy, y^2+x^3-2xy^3\right\rangle \\ &=\left\langle xy, x^3+y^2\right\rangle. \end{aligned}$$

For $\left\langle xy, x^3+y^2\right\rangle$, the only null point of

$$\begin{cases} z_1 z_2=0 \\ z_1^3+z_2^2=0 \end{cases}$$

is $z_1=z_2=0$ in complex field, I is codimensional in E_2 by virtue of theorem 3.

One will find a team of base of the complementary space of finite codimensional ideal I in E_2.

Because $\overline{I_0\cap P_2^0}\oplus\overline{I_1\cap P_2^1}=R\{1,x,y\}$, for the standard base $\{1\}$ of P_2^0 and $\{x,y\}$ of P_2^1, $1, x, y$ are part of a team of base of the complementary space of I in E_2.

For the standard base $\{x^2, xy, y^2\}$ of P_2^2, $x^2\notin I\oplus R\{1,x,y\}$, $x^2\in\overline{I_2\cap P_2^2}$, so x^2 belongs to the base of $\overline{I_2\cap P_2^2}$, as well as x^2 belongs to a team of base of the complementary space of finite codimensional ideal I in E_2.

One can add x^2 to the base of the complementary space of ideal I, then I and its complementary space will be expanded to $I\oplus R\{1,x,y,x^2\}$ in E_2.

Moreover, $xy\in I\subset I\oplus R\{1,x,y,x^2\}$, then $xy\notin\overline{I_2\cap P_2^2}$, xy does not belong to the base of $\overline{I_2\cap P_2^2}$ and the complementary space of finite codimensional ideal I in E_2.

In addition, according Lemma 3, whether y^2 belong to $I\oplus R\{1,x,y,x^2\}$ depends on the subordinate relationship between x^3 and I.

Remark: y^2 and x^3 are the part of generators of $I=\left\langle xy, x^3+y^2\right\rangle$. One deduce that $x^3\notin I$, if not, there exist $\eta_1(x,y)$ and $\eta_2(x,y)\in E_2$ in E_2 such that

$$x^3=\eta_1(x,y)(x^3+y^2)+\eta_2(x,y)xy.$$

Suppose that $x = 0$, then

$$\eta_1(0, y)y^2 = 0,$$
$$\eta_1(x, y) = x\eta_1^1(x, y),$$
$$x^3 = x\eta_1^1(x, y)(x^3 + y^2) + \eta_2(x, y)xy,$$
$$x^2 = \eta_1^1(x, y)(x^3 + y^2) + \eta_2(x, y)y.$$

One deduce that

$$x^2 \in \left\langle y, x^3 + y^2 \right\rangle = \left\langle y, x^3 \right\rangle.$$

That is contradictory. It is impossible that

$$x^3 \in I = \left\langle xy, x^3 + y^2 \right\rangle.$$

By virtue of this inference, one get $y^2 \in \overline{I_2 \cap P_2^2}$, so y^2 belongs to a team of base of the complementary space of I in E_2. One can add y^2 to the base of the complementary space of finite codimensional ideal I, then I and its complementary space will be expanded to $I \oplus R\{1, x, y, x^2, y^2\}$ in E_2.

For the standard base $\{x^3, x^2y, xy^2, y^3\}$ of P_2^3, between the generators of $I = \left\langle xy, x^3 + y^2 \right\rangle$, only $y^2 + x^3$ can generate x^3. The term of $y^2 + x^3$ those power less than 3 excludes x^3 due to Lemma 3, then

$$x^3 \in I \oplus R\{1, x, y, x^2, y^2\}.$$

With the same reason, one can infer that

$$x^2 y \in I \subset I \oplus R\{1, x, y, x^2, y^2\},$$
$$xy^2 \in I \subset I \oplus R\{1, x, y, x^2, y^2\}.$$

For y^3, only $y^2 + x^3$ can generate y^3 for the generators of $I = \left\langle xy, x^3 + y^2 \right\rangle$. Because

$$y(y^2 + x^3) = y^3 + x^3 y,$$

one can infer $x^3 y \in I$, and then

$$y^3 \in I \oplus R\{1, x, y, x^2, y^2\}.$$

One knows $I \supset M^4$, it is no use to inspect the standard base of P_2^4.

In conclusion, $I \oplus R\{1, x, y, x^2, y^2\} = E_2$, therefore a team of base of the complementary space of finite codimensional ideal I in E_2 is $\{1, x, y, x^2, y^2\}$.

3. CONCLUSIONS

The problem of finite codimensional ideals is always the core issue in the study of singularity theory. However, relevant studies rarely refer to determining a basis set of the complementary space of a finite codimensional ideal and the judgment of finite codimensional ideals in E_n. In this study, these two problems were studied by using specific algebraic knowledge, and specific examples were presented. Calculation of the complementary space of finite codimensional ideals in the complex analytic ring of germs θ_n will be the direction of future research.

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