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REVERSE DERIVATIONS IN PRIME RINGS WITH RIGHT IDEALS

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ABSTRACT

In this paper we present some results on the reverse derivations in prime rings with right ideals. We prove that if a reverse derivation d acts as a homomorphism or an antihomomorphism on a nonzero right ideal U of a prime ring R, then d = 0. Also, we show that if [d(x), x] = 0 or [d(x), d(y)]=0 or [d(x), d(y)] = [x,y] for all x,y ∈U, then R is commutative.

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INTRODUCTION

Mecdonald [3] established some group-theoretic results in terms of inner derivations. Bell and Kappe [1] studied the analogous results for rings in which derivations satisfy certain algebraic conditions. Bell and Moson [2] proved the commutativity of near-rings and rings using strong commutativity-preserving derivations. We prove that if a reverse derivation d acts as a homomorphism or an antihomomorphism on a nonzero right ideal U of a prime ring R, then $d = 0$. Also, we show that if $[d(x), x] = 0$ or $[d(x), d(y)] = 0$ or $[d(x), d(y)] = [x, y]$ for all $x, y \in U$, then R is commutative.

PRELIMINARIES

Throughout this paper R will denote a prime ring and Z its Centre. A ring R is prime if whenever A and B are ideals of R such that $AB = 0$ then either $A = 0$ or $B = 0$. Also a ring R is called prime if $xay=0$ implies $x = 0$ or $y = 0$ for all x, y, a in R. A ring R is said to be n-torsion free, if there exists a positive integer n such that $nx = 0$ implies $x = 0$ for all $x \in R$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $d : R \to R$ is a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. We use the identities

$$[xy, z] = [x, z]y + x[y, z],\ [x, yz] = [x, y]z + y[x, z]$$

To prove the main results we require the following results [1]:

Lemma 1:

(i) Let U be a subring of a ring R and let d be a derivation of R which acts as a homomarphism on U. Then $d(x)x(y\text{-}d(y)) = 0$ for all $x, y \in U$.

(ii) Let V be a right ideal of R and d be a derivation of R acting as an antihomomorphism of V. Then $d(x)y[r, d(x)]=0$ for all $x, y \in V$ and $r \in R$.

Theorem 1: Let R be a semiprime ring. If d is a derivation of R which is either an endomorphism or an antiendomorphism, then $d = 0$.

Theorem 2: Let R be a prime ring and U a nonzero right ideal of R. If d is a derivation of R which acts as a homomarphism or an antihomomorphism on U, then $d = 0$ on R.

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Now we prove the following results:

Theorem 3: Let R be a prime ring and U a nonzero right ideal of R. Suppose d: $R \to R$ is a reverse derivation of R,

(i) If d acts as a homomorphism on U, then $d = 0$ on R.

(ii) If d acts as an antihomomorphism on U, then $d = 0$ on R.

Proof: (i) If d acts as a homomorphism on U, then we have

$d(y)\, d(x) = d(yx) = d(x)y + xd(y)$, for all $x, y \in U$. (1)

We replace $y = yx$ in equation (1), then

$d(yx)d(x) = d(x)yx + xd(yx)$, for all $x, y \in U$. (2)

By multiplying (1) with $d(x)$ on right side and using d is a homomorphism on U, we get

$d(yx)d(x) = d(x)\, yd(x) + xd(y)d(x)$.

$d(yx)d(x) = d(x)\, yd(x) + xd(yx)$ (3)

By combining equations (2) and (3), we get

$d(x)yx = d(x)yd(x)$, for all $x, y \in U$ (4)

i.e., $x = d(x)$.

So, $(d(x) - x)\, d(x) = 0$.

Thus $d\ (x^2) = xd(x)$.

Since d is a reverse derivation, we have $d(x)x = 0$.

By linearizing x, we obtain

$d(x)y + d(y)x = 0$, for all $x, y \in U$. (5)

We replace y by xy in equation (5), then we have

$d(y)\, xx = 0$, for all $x, y \in U$ (6)

If we right multiply by x in equation (5), we get

$d(x)yx + d(y)xx = 0$, for all $x, y \in U$.

From the above equations, we obtain

$d(x)yx = 0$, for all $x, y \in U$.

By substituting y by ys in this equation, we get $d(x)\ ysx = 0$, for all $x,y \in U$ and $s \in R$. Thus for each $x \in U$, the primeness of R implies that either $d(x)y$=0 or x=0. But $x = 0$ also implies that

$d(x)y = 0$, for all $x, y \in U$. (7)

If we replace x by xr in equation (7), we get

$d(xr)y = 0$, for all $x,y \in U$ and $r \in R$.

Then $d(r)xy + rd(x)y = 0$. So we get

$d(r)xy = 0$, for all $x,y \in U$ and $r \in R$ (8)

Again we replace x by xs in equation (8). We have

$d(r)xsy = 0$, for all $x, y \in U$ and $s, r \in R$.

i.e. $d(r)\ x\ R\ y = 0$, for all $x, y \in U$ and $s, r \in R$.

Since R is prime, it follows that

$d(r)x = 0$, for all $x, y \in U$ and $r \in R$. (9)

In equation (9), we substitute r by rs. Then we have

$d(rs)x = 0$ for all $x \in U$ and $r, s \in R$

i.e. $d(s)rx + sd(r)x = 0$, for all $x \in U$ and $r, s \in R$. So we get

$d(s)rx = 0$, for all $x \in U$ and $r, s \in R$. (10)

i.e., $d(s)R\ x = 0$, for all $x \in U$ and $r, s \in R$.

Since R is prime, either $d(s) = 0$ or $x = 0$. But $x = 0$ also implies that $d(s) = 0$, for all $s \in R$, then $d = 0$ on R.

(ii) Suppose d acts as an antihomomorphism on U. By our hypothesis, we have

$$d(xy) = d(y)\,d(x) = d(y)x + y\,d(x), \text{ for all } x, y \in U. \tag{11}$$

By substituting y by xy in equation (11), then

$$d(xy)d(x) = d(x(xy)), \text{ for all } x, y \in U.$$
$$= d((xx)y)$$

$$d(xy)\,d(x) = d(y)xx + yd(xx), \text{ for all } x, y \in U. \tag{12}$$
$$d(xy)\,d(x) = d(y)x\,d(x) + y\,d(x)\,d(x), \text{ for all } x, y \in U \tag{13}$$

By combining equations (12) and (13). Then

$$d(y)x\,d(x) = d(y)xx, \text{ for all } x, y \in U. \tag{14}$$

i.e. $d(x) = x$, for all $x \in U$.

So $(d(x) - x) = 0$, for all $x \in U$.

We right multiply this equation with $d(x)$. Then

$$(d(x) - x)\,d(x) = 0, \text{ for all } x \in U.$$

Thus $d(x^2) = x\,d(x)$, for all $x \in U$.

Since d is a reverse derivation, we have $d(x)\,x = 0$.

By linearazing x, we obtain

$$d(x)y + d(y)\,x = 0, \text{ for all } x, y \in U. \tag{15}$$

We replace y by xy in equation (15), then we get $d(y)xx = 0$. So, we have obtained equation (6). The remaining proof is same as in proof of (i).

Theorem 4: Let R be a 2-torsim free prime ring, U a nonzero right ideal of R and d be a nonzero reverse derivation of R. If $[d(x),x] = 0$ for all $x \in U$, then R is commutative.

Proof: We have $[d(x), x] = 0$ for all $x \in U$. (16)

By linearizing x, in equation (16), we obtain

$$[d(x), y] + [x, d(y)] = 0, \text{ for all } x, y \in U. \tag{17}$$

By substituting y with yx in equation (17), we get

$$[d(x), yx] + [x, d(yx)] = 0, \text{ for all } x, y \in U.$$
$$[d(x), y]x + y[d(x), x] + [x, d(x)y] + [x, xd(y)] = 0, \text{ we have}$$
$$[d(x), y]x + [x, d(x)]y + d(x)\,[x, y] + [x, x]\,d(y) + x[x, d(y)] = 0,$$

then we get

$$d(x)[x, y] = 0, \text{ for all } x, y \in U. \tag{18}$$

We replace y by yz in equation (18), we have

$$d(x)\,[x, yz] = 0, \text{ for all } x, y, z \in U. \text{ We get}$$
$$d(x)\,[x, y]z + d(x)y\,[x, z] = 0, \text{ then}$$
$$d(x)y\,[x, z] = 0,\ x, y, z \in U \text{ according to (18).}$$

Again by substituting y by yr in this equation, we have

$$d(x)\,yr[x, z] = 0, \text{ for all } x, y, z \in U \text{ and } r \in R.$$

Since R is prime, either $d(x)y = 0$ or $[x, z] = 0$. If $d(x)y = 0$, then $d(U)U=\{0\}$.

But $d(U)U \neq \{0\}$, since $d \neq 0$, $U \neq 0$ and R is prime. Thus $[x, z] = 0$ for all $x, z \in U$. So U is commutative.

Hence R is commutative.

Theorem 5: Let R be a 2-torsion free prime ring, U be a nonzero right ideal of R and d be a nonzero reverse derivation of R. If $[d(x), d(y)] = 0$ for all $x,y \in U$, then R is commutative.

Proof: we have $[d(x), d(y)] = 0$. (19)

By taking $y = yx$ in equation (19), we have

$[d(x), d(yx)] = 0$, for all $x, y \in U$.

$[d(x), d(x)y + xd(y)] = 0$.

$[d(x), d(x)y] + [d(x), d(x)]y + x[d(x), d(y)] + [d(x),x]\, d(y) = 0$. We get

$d(x)\, [d(x),y] + [d(x),x]d(y) = 0$ for all $x, y \in U$. (20)

By substituting $d(y)$ with $d(z)y$ in equation (20), we have

$d(x)\, [d(x),y] + [d(x),x]\, d(z)y = 0$, for all $x, y, z \in U$. (21)

Again we take y by yr in equation (21). Then we have

$d(x)[d(x), yr] + [d(x), x]\, d(z)\, yr = 0$, for all $x, y, z \in U$ and $r \in R$.

$d(x)y\, [d(x),r] + d(x)[d(x),y]r + [d(x),x]\, d(z)\, yr = 0$. (22)

From equations (21) and (22), we get

$d(x)y[d(x),r] = 0$, for all $x, y, z \in U$ and $r \in R$.

$d(x)\, U\, [d(x),r] = \{0\}$.

$d(x)\, U\, R[d(x),r] = \{0\}$.

Since R is prime we have either $d(x)\, U = \{0\}$ or $[d(x), r] = 0$.

Since $d \neq 0$, $U \neq \{0\}$ and R is prime it follows that $d(x)\, U \neq 0$.

So $[d(x), r] = 0$. Then $d(x) \in Z$, centre of R. Hence $[d(x), x] = 0$, for all $x \in U$.

From Theorem 4, R is commutative.

Theorem 6: Let R be a 2-torsion free prime ring, U be a nonzero right ideal of R and d be a nonzero reverse derivation of R. If $[d(x), d(y)] = [x, y]$ for all $x, y \in U$, then R is commutative.

Proof: We have $[x, y] = [d(x), d(y)]$, for all $x, y \in U$. (23)

By taking y by yz in the equation (23), we have

$[x, yz] = [d(x), d(yz)]$

$y[x, z] + [x, y]z = [d(x), d(z)y + zd(y)]$.

$y[x, z] + [x, y]z = [d(x), d(z)y] + [d(x), zd(y)]$.

$y[x, z] + [x, y]z = d(z)\, [d(x), y] + [d(x), d(z)]y + z[d(x), d(y)] + [d(x),z]d(y)$.

From Lemma ([2] Lemma 5(ii)), we obtain

$d(z)[d(x), y] + [d(x), z]d(y) = 0$. (24)

We put $z = x$ in this equation. Then

$d(x)[d(x), y] + [d(x),x]d(y) = 0$. This is equation (20). The remaining proof is similar to the proof of Theorem 5.

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