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COMMON FIXED POINT THEOREM IN CONE 2 – METRIC SPACE FOR CONTRACTIVE MAPPING

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ABSTRACT

In this paper we have prove some common fixed point theorems for contractive mappings in the setting of Cone 2-metric spaces. Our results improve, extend and generalize significant recent results of S.K. Tiwari et al. [17].

Key words: *Cone metric space, Cone metric space, contractive mappings, Common fixed point.*

1. INTRODUCTION

Fixed point theory is one of the most dynamic research subject in nonlinear analysis. The theory itself is a beautiful mixture of analysis, topology and geometry .over the last years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of non- linear phenomena. In this area, the first important and significant results was prove by banach [1] in 1922 for a contraction mapping in a complete metric space. The well knows banach contraction theorems may be stated as follows:

"Every contraction mapping of a complete metric space in X.

The Banach fixed point theory is important as a source of existence and uniqueness theorem in different branches on analysis. In this way theorem provides an impressive illustration of the unifying power of functional analytic method and of the usefulness of fixed point theorems in analysis.

The study of fixed points of mappings satisfying certain contractive conditions has been very active area of research. Recently Long-Guang and Xian [9] generalized the concept of a metric space, by introducing cone metric spaces, and obtained some fixed point theorem for mappings satisfying certain contractive conditions. One can consider a generalization of a cone metric space by replacing the triangle inequality by a more general inequality. As such, every cone metric is a generalized cone metric space but the converse is not true The Banach fixed point theory or contraction theorem concerns certain mapping of a complete metric space into itself. It states sufficient condition for the existence and uniqueness of a fixed point (point that is mapped onto itself). The theorem also gives an iterative process by which we can obtain approximation to the fixed point and error bounds.

Motivated by this work, several authors introduced similar concepts and prove analogous fixed point theorem in 2 – metric and 2 – Banach space. Gahler ([2], [3], and [4]) investigated the concept of 2 – metric space and give the definition as follows:

Definition 1.1: Let X be a non empty set and let $d: X \times X \times X \to R:, i.e. X^3$→R satisfying the following condition
a. $d(x,y,z) = 0$ if at least two of x,y,z are equal,
b. $d(x,y,z) = d(p(x,y,z))$ $for\ all\ x,y,z \in X$ and for all permutations $p(x,y,z)$ $of\ x,y,z.$
c.$d(x,y,z) \leq d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all $x,y,z,w \in X$. Then (X,d) is called a 2 – metric space which sometimes be denoted simply by X, when there is no confusion it can easily seen that d is non–negative function. Perhaps lseki [5-7] obtained for the first time basic results on fixed point of operators in 2 – metric space and in 2 – Banach space. After the work of iseki, several authors extended and generalized fixed point theorem in 2 – metric and 2- banach space for different type of operators of contractive type.

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Recently, L.G. Huang and X. Zhang [8 introduced cone metric space by generalized the concept of a metric space, replacing the set of real numbers, by an ordered Banach space and obtained some fixed point theorems for contractive mappings. By using both of the concepts, we define a new space, Cone 2-metric space by replacing real number in 2-metric space by an ordered Banach space.

Subsequently, many authors have studied the strong convergence to a fixed point with contractive constant in cone metric space, see for instance ([9], [10], [11], [12], [13], [14].) on the other hand S. Singh, S. Jain, and Bhagat [15] introduce cone 2- metric space by replacing real number in 2 – metric space by an ordered Banach space and some fixed point theorem for contractive mapping on complete cone 2 – metric space with assumption of normality on the cone. In sequel, S.K. Tiwari *et al.* [17] extend and prove fixed point results for contractive mappings in setting of cone 2- metric spaces, which is generalize of the result [15].

The purpose of this paper is to extend, improves and generalize the common fixed point theorems of S.K. Tiwari *et al.* [17].

2. PRELIMINARIES

First we recall some standard notation and definition in cone metric space with some of their properties [8]

Definition 2.1: Let E be a real Banach space and P a subset of $E. P$ is called a cone if
1. P is closed, nonempty and $P \neq \{0\}$.
2. $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$.
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \leq y$ will stand for $y - x \in intP$, intP denotes the interior of P.

The cone is called normal if there exists a number K > 0 such that ∀x, y∈ P.

$x \leq y$ implies $||x|| \leq K||y||$

The least number satisfying above is called the normal constant of P. The cone P is called regular if every non-decreasing sequence in P, which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$x_1 \leq x_2 \leq \cdots \leq x_n \dots \dots . \leq y$

for some y in E, there exists $x \in P$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It can be easily proved that a regular cone is a normal cone.

Throughout this paper, we suppose that E is a real Banach space, P is cone in E with $intP \neq \varphi$ and is partial ordering in E $w$ith respect to P.

Remark 2.2 [16]: If E is a real Banach space with cone P and if a ≤λa where $a \in P$ and 0 < λ < 1, then $a = 0$.

Definition 2.3[15]: Let X be a non empty set. Suppose the mapping $d: X \times X \times X \to X$ satisfies the following condition;
1. $0 \leq d(x, y, z)$ $\forall x, y, z \in X$ and $d(x, y, z) = 0$ iff at least two of x, y, z are equal
2. $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and ∀ permutation $p(x, y, z)$ of x, y, z
3. $d(x, y, z)l \leq d(x, y, w) + d(x, w, z) + d(w, y, z), \forall$ $x, y, z, w \in X$ then d is called a cone 2- metric on X, and (X, d) will be called a cone 2-metric space.

Example 2.4[15]: Let $E = R^2, p\{(x, y) \in E | x, y \leq 0\} \Longrightarrow R^2$

$X = R$ and $X \times X \times X \to E$ and such that d(x, y, z)= $(\rho^n \propto \rho)$ where $\rho = min(|x - y|, |y - z|, |z - x|)$ and $\propto$ and n are some fixed positive integer. Then (X, d) is a cone 2- metric space.

Definition 2.5[15]:
- (i) Let (X, d) be a cone 2-metric space with respect to a cone P in a real Banach space E. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with$0 \ll c$ $i.e.$ $c \in intP$ there is N such that $d(x_n, x, a) \ll$c for all $a \in X$ and for all $n > N$. Then $\{x_n\}$ is said to be convergent to x. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
- (ii) Let (X, d) be a cone 2-metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ there is a N such that $d(x_n, x_m, a) \ll c$ for all $a \in X$ and for all $m, n > N$, then $\{x_n\}$ is said to be a Cauchy sequence in X.
- (iii) Let (X, d) be a cone 2-metric space, if every Cauchy sequence is convergent in X, the X is said to be a complete cone 2-metric space.

Lemma 2.6:

(i) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x, a) \to 0$ as $n \to\infty$ for all $a \in X$.

(ii) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then limit of $\{x_n\}$ is unique if it exists.

(iii) Let (X, d) be a cone 2-metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

(iv) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, y_n, a) \to 0, (n, m \to\infty)$, for all $a \in X$.

(v) Let (X, d) be a cone 2-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, $x_n \to x, y_n \to y\ (n \to \infty)$. Then $d(x_n, y_n, a) \to d(x, y, a)\ (n \to \infty)$ for all $a \in X$.

Definition 2.7: Let (X, d) be a cone 2-metric space. If for every sequence $\{x_n\}$ in X, there is a subsequence $\{x_{ni}\}$ of $\{x_n\}$ converging in X. Then X is called a sequentially compact cone 2-metric space

3. MAIN RESULTS

In this section we shall prove some fixed point theorems for contractive maps by using normality of the cone.

Theorems 3.1: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping $T_1, T_2: X \to X$ satisfy the contractive condition.

$$d(T_1x, T_2y, a) \le a_1 d(x, y, a) + a_2 d(T_1x, x, a) + a_3 d(T_2y, y, a) + a_4 d(T_2y, x, a) + a_5 d(T_1x, y, a)$$

for all $x, y \in X$. and ai , $i = 1,2,3,4,5$, are all non negative constant with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. then T_1 and T_2 have a unique common fixed point in X, for all $x \in X$. iterative sequence $\{{T_1}^{2n+1}x\}$ and $\{{T_2}^{2n+2}x\}$ is conversant to common fixed point.

Proof: Let $x_0 \in X$ be fixed point.

Let $x_1 = Tx_0, x_3 = T_1x_2 = T^3{}_1 x_0 \dots \dots.$

$$x_{2n+1} = Tx_{2n} = T^{2n+1}x_0 \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Similarly, we have $x_2 = T_2x_1, x_3 = {T_2}^2x_0, x_4 = T_2x_3 = {T_2}^4x_0, \dots\dots\dots\dots\dots$

$$X_{2n+2} = Tx_{2n+1} = T^{2n+2}x_0 \dots\dots\dots\dots\dots\dots\dots\dots\dots.$$

From (2.1) Taking $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}, a) &= d(T_1x_{2n}, T_2x_{2n-1}, a) \\ &\le a_1 d(x_{2n}, x_{2n-1,} a) + a_2 d(T_1x_{2n}, x_{2n}, a) + a_3 d(T_2x_{2n-1}, x_{2n-1}, a) \\ &+ a_4 d(T_2x_{2n-1}, x_{2n}, a) + a_5 d(T_1x_{2n}, y_{2n-1}, a) \\ &\le a_1 d(x_{2n}, x_{2n-1,} a) + a_2 d(x_{2n+1}, x_{2n}, a) + a_3 d(x_{2n}, x_{2n-1}, a) \\ &+ a_5 [d(x_{2n}, x_{2n-1}, a) + d(x_{2n}, x_{2n+1}, a)] \\ &\le (a_1 + a_3 + a_5) d(x_{2n}, x_{2n-1,} a) + (a2 + a5)\, d(x_{2n}, x_{2n+1}, a) \end{aligned}$$

This implies

$$(1 - a_2 - a_5)\, d(x_{2n+1}, x_{2n}, a) \le (a_1 + a_3 + a_5)\, d(x_{2n}, x_{2n-1}, a)$$

$$\Rightarrow \quad d(x_{2n+1}, x_{2n}, a) \le Ld(x_{2n}, x_{2n-1}, a) \quad Where\ L = \frac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)} \tag{2}$$

Hence

$$\begin{aligned} &\le L^2\{d(x_{2n-1,}, x_{2n-2}, a)\} \\ &\le \cdots\dots\dots\dots\dots\dots \le L^n\{F(x_1, x_0\ , a)\} \end{aligned} \tag{3}$$

Also for $k > t$, we have

$$\begin{aligned} d(x_{2k}, x_{2k-1}, x_{2t}) &\le Ld(x_{2k-1}, x_{2k-2}, x_{2t}) \\ &\le L^2 d(x_{2k-1}, x_{2k-2}, x_{2t}) \\ &\le \cdots\dots\dots\dots\dots\dots\dots\dots\dots. \le \\ &\le L^{2k-2t-1} d(x_{2t+1}, x_{2t}, x_{2t}) \\ &= 0 \end{aligned} \tag{4}$$

Now for $n > m$, with using (3) and (4), we have

$$\begin{aligned} d(x_{2n}, x_{2m}, a) &\le d(x_{2n}, x_{2m}, x_{2n-1}) + d(x_{2n}, x_{2n-1}, a) + d(x_{2n-1}, x_{2m}, a) \\ &\le L^{2n-1} d(x_1, x_0, a) + d(x_{2n-1}, x_{2m}, x_{2n-2}) \\ &+ d(x_{2n-1}, x_{2n-2}, a) + d(x_{2n-2}, x_{2n-3}, a) + d(x_{2n-3}, x_{2m}, a) \end{aligned}$$

$$\leq (L^{2n-1} + L^{2n-2} + L^{2n-3})\, d(x_1, x_0, a) + d(x_{2n-3}, x_{2m}, a)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leq (L^{2n-1} + L^{2n-2} + \cdots \ldots\ldots\ldots + L^{2m+1})\, d(x_1, x_0, a) + d(x_{m+1}, x_m, a)$$
$$\leq (L^{2n-1} + L^{2n-2} + \cdots \ldots\ldots\ldots + L^{2m+1} + L^m)\, d(x_1, x_0, a)$$
$$= L^{2m}(1 + L + L^2 + L^3 + \ldots\ldots L^{2n-2m-1})\, d(x_1, x_0, a)$$
$$\leq \left(\frac{L^{2m}}{1-L}\right) d(x_1, x_0, a) \text{ as } l< 1 \text{ and } P \text{ is closed.}$$

Thus we have

$$||\, d(x_{2n}, x_{2m}, a)|| \leq \left(\frac{L^{2m}}{1-L}\right) ||\, d(x_1, x_0, a)||$$

T$his\ implies (d(x_1, x_0, a) \rightarrow 0, (n, m \rightarrow \infty),\ for\ all\ a \in X$.

Hence $\{x_n\}$ is a Cauchy sequence. in (X, d) is a complete cone 2 –metric space, then there exist, $u \in X$ such that $x_{2n} \rightarrow (n \rightarrow \infty.).\ lim\ n \rightarrow \infty x_{2n} = u$.Now for $a \in X$, then we have

$$d(T_1 u, u, a) \leq d\,(T_1 u, u, Tx_{2n}) + d(T_1 u, T_1 x_{2n}, a) + d(T_1 x_{2n}, u, a)$$
$$\leq a_1 d(u, x_{2n}, u) + a_2 d(T_1 u, u, a) + a_3 d(Tx_n, x_n, a)$$
$$+ a_4 d(Tx_{2n}, u, a) + a_5 d(T_1 u, x_n, a) + d(x_{2n+1}, u, a)$$
$$\leq a_1 d(u, x_{2n}, u) + a_2 d(T_1 u, u, a) + a_3 d(Tx_{2n+1}, x_{2n}, a)$$
$$+ a_4 d(Tx_{2n+1}, u, a) + a_5 d(T_1 u, x_{2n}, a) + d(x_{2n+1}, u, a)$$
$$\leq a_1 d(u, x_{2n}, u) + a_2 d(T_1 u, u, a) + a_3 [d(u, x_{2n}, a) + d(x_{2n+1}, u, a)]$$
$$+ a_4 d(x_{2n+1}, u, a)] + a_5 [d(u, x_{2n}, a) + d(T_1 u, u, a)] + d(x_{n+1}, u, a)$$

$$(1 - a_2 - a_5)\, d(T_1 u, u, a) \leq (a_1 + a_3 + a_5)\, d(u, x_{2n}, a) + (a_3 + a_5) d(u, x_{2n+1}, a) + d(x_{n+1}, u, a)$$
$$\Rightarrow d(T_1 u, u, a) \leq \left(\frac{a_1 + a_3 + a_5}{1 - a2 - a5}\right) d(u, x_{2n}, a) + \frac{(a_3 + a_5)}{(1 - a_2 - a_{5)}} d(x_{2n+1}, x_{2n}, a)$$
$$+ \frac{1}{(1 - a_3 - a_4)} + d(x_{2n+1}, u, a) \rightarrow 0$$

On taking limit as n→∞ and by using 2.6, we get

$$d(T_1 u, u, a) = 0.$$

This implies that $T_1 u = u$. So u is a fixed point of T_1 in X .Now if v is another fixed point of T_1 in, then

$$d(u, v, a) = (Tu, Tv, a)$$
$$\leq a_1 d(u, v, a) + a_2 d(T_1 u, u, a) + a_3 d(T_2 v, v, a) + a_4 d(T_2 v, u, a) + a_5 d(T_1 u, v, a)$$
$$= (a_1 + a_3 + a_5) d(u, v, a)$$

By using remark 2.2 we obtain that $d(u, v, a) = 0$.and thus $u = v$. therefore the fixed point of $T_1$is unique.

Similarly it can be established that $T_2 u = u$. Hence $T_1 u = u = T_2 u$. Thus u is common fixed point of T_1 and T_2 This completes the proof.

On taking $a_1 = k$ and $a_2 = a_3 = a_4 = a_5 = 0$ in theorem 3.1, we get the following corollary in the setting of cone 2 –metric space.

Corollary 3.2: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping $T_1, T_2: X \rightarrow X$ satisfy the contractive condition

$$d(T_1 x, T_2 y, a) \leq k d(x, y, a)$$

for all $x, y, a \in X$ where $k \in [o, 1]$ is a constant. then This is a unique fixed point in X .And for any $x \in X$, iterative sequence $\{T_1^{2n+1} x\}$ and $\{T_2^{2n+2} x\}$ is conversant to fixed point.

Corollary 3.3: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping $T_1, T_2: X \rightarrow X$ satisfy the contractive condition

$$d(T_1 x, T_2 y, a) \leq \alpha [d(T_1 x, x, a) + d(T_1 y, y, a)] + \beta [d(T_1 y, x, a) + d(T_1 x, y, a)]$$

$for\ all\ x, y, a \in X$ and $\alpha, \beta \in [0,1]$ are all non negative constant $with\ \alpha + \beta_1$. then T_1 and T_2 have an unique common fixed point in X. and for $any\ x \in X$, iterative sequence $\{T_1^{2n+1} x\}$ and $\{T_2^{2n+2} x\}$ is conversant to fixed point.

Proof: Putting $a_1 = 0, a_2 = a_3 = \alpha\ and\ a_4 = a_5 = \beta$ in theorem 3.1.we get the required results easily.

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Corollary 3.4: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping $T_1\ , T_2: X\ \to X$ satisfy the contractive condition.
$d(T_1x, T_2y, a)\ \leq a_1d(x, y, a) + a_2d(T_1x, x, a) + a_3d(\ T_2y, y, a) +\ a_4[d(\ T_2y, x, a) + d(T_1x, y, a) for\ all\ x, y, a \in X$ and $a_i, i = 1,2,3,4,$. Are all non negative constant with $a_1 + a_2 +\ a_3 + a_4 + a_5 < 1$. then $T_1$and T_2 have an unique common fixed point in X .and for $any\ x \in X$,iterative sequence $\{T_1^{2n+1}x\}$ and$\{T_2^{2n+2}x\}$ is conversant to fixed point.

Proof: Putting $a_4 = a_5$ a in theorem 3.1 , we get the required results easily.

Corollary 3.5: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping $T_1, T_2: X\ \to X$ satisfy the contractive condition

$d(T_1x, T_2y, a)\ \leq \alpha d(x, y, a) + \beta[d(T_1x, x, a) + d(\ T_2y, y, a)]$ for all $x, y, a \in X$. and $\alpha, \beta \in [0,1]$ are all non negative constant $\alpha + 2\beta\ < 1$. then $T_1$and T_2 have an unique common fixed point in X .and for $any\ x \in X$, iterative sequence $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ is conversant to fixed point.

Proof: Putting $a_1 = \alpha, a_2 =\ a_3 = \beta, a_4 = a_5 = 0$ in theorem 3.1, then we get required results easily.

Theorems 3.6: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant K. Suppose the mapping $T_1\ , T_2: X\ \to X$ satisfy the contractive condition.

$$d(T_1x, T_2y, a)\ \leq a_1d(x, y, a) + a_2d(x,T_1x, a) + a_3d(y, T_2y, a)+a_4[d(x, T_2y, a) + d(y, T_1x, a)] \quad (3.6.1)$$

$for\ all\ x, y \in X. And\ a_i, i = 1,2,3,4, . Are\ all\ non\ negative\ constant\ with\ a_1 + a_2 + a_3 + a_4 + < 1.\ then\ T_1$ and $T_2\ \ have\ unique\ fixed\ point\ in\ X. and\ for\ all\ x \in X. Iterative\ sequence\ \{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}\}$ is conversant to fixed point.

Proof: Let $x_0 \in X$ be fixed point. $Let\ x_1 = Tx_0, x_3\ = T_1x_1\ = T_1^3x_0 \dots \dots \dots x_{2n+1} = Tx_{2n}\ = T^{2n+1}x_0$ ....

Similarly, we have $x_2\ = T_1x_1, x_3 = T^2x_0\ , x_4\ = T_2x_3 = T_2^4x_0 \dots \dots x_{2n+2} = Tx_{2n+1} = T^{2n+2}x_0$

From (3. 6) Taking $x = x_{2n}$ and y = x_{2n-1}, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}\ , a) &= d(T_1x_{2n}, T_2x_{2n-1}\ , a) \\ &\leq a_1d\big(x_{2n}, x_{2n-1,}a\big) + a_2d(T_1x_{2n}, x_{2n}, a) + a_3d\big(x_{2n-1\ ,}T_2x_{2n-1}, a\big) \\ &\quad +a_4[d(x_{2n}, T_2x_{2n-1}, a) + d(x_{2n-1\ ,}T_1x_{2n}, a)\ ] \\ &\leq a_1d\big(x_{2n}, x_{2n-1,}a\big) + a_2d\big(x_{2n}, x_{2n+1\ ,}a\big) + a_3d(x_{2n-1}x_{2n}, a) \\ &\quad + a4[[d(x_{2n}, x_{2n-1,}a)\ + d(x_{2n}, x_{2n+1\ ,}a)] \\ &\leq (a_1 +\ a_3 +\ a_4\ )d(x_{2n}, x_{2n-1,}a)\ +\ (a_2 +\ a_4)\ d(x_{2n+1\ ,}x_{2n}a) \end{aligned}$$

$$(1 - a_2 -\ a_4)\ d(, x_{2n+1\ ,}x_{2n}a) \leq (a_1 +\ a_3 +\ a_4)\ d(x_{2n}, x_{2n-1,}a)$$

$$d(x_{2n+1\ ,}x_{2n}a))\ \leq Ld(x_{2n}, x_{2n-1,}a)$$

$$Where\ L = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_4} \quad (3.6.2)$$

$$\leq L^2\{d(x_{2n-1,}, x_{2n-2}, a)\}$$

$$\leq \cdots \dots \dots \dots \dots \dots \leq\ L^n\{F(x_1, x_0 \quad (3.6.3)$$

Also for $k > t$, we have

$$\begin{aligned} d(x_{2k}, x_{2k-1}, x_{2t}) &\leq Ld(x_{2k-1}, x_{2k-2}, x_{2t}) \\ &\leq L^2d(x_{2k-2}, x_{2k-3\ ,}x_{2t}) \\ &\leq \cdots \dots \dots \dots \dots \dots \dots \dots \dots . \leq \\ &\leq L^{2k-2t-1}d(\ x_{2t+1}, x_{2t}, x_{2t}) \\ &= 0 \end{aligned} \quad (3.6.4)$$

Now for $n > m$, with using (3.63) and (3.6.4) , we have

$$\begin{aligned} d(x_{2n}, x_{2m}, a) &\leq d(x_{2n}, x_{2m}, x_{2n-1}) +\ d(x_{2n}, x_{2n-1}, a) + d(x_{2n-1}, x_{2m}, a) \\ &\leq L^{2n-1}d(x_1, x_0, a) + d(x_{2n-1}, x_{2m}, x_{2n-2}) + d(x_{2n-1}, x_{2n-2}, a) \\ &\quad + d(x_{2n-2}, x_{2n-3}, a) +\ d(x_{2n-3}, x_{2m}, a) \\ &\leq (L^{2n-1} + L^{2n-2}\ + L^{2n-3})\ d(x_1, x_0, a) + d(x_{2n-3}, x_{2m}, a) \\ &\qquad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ &\qquad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ &\leq (L^{2n-1} + L^{2n-2} + \cdots \dots \dots \dots \dots . +L^{2m+1}\ )d(x_1, x_0, a) + d(x_{m+1}, x_m, a) \\ &\leq\ (L^{2n-1} + L^{2n-2} + \cdots \dots \dots \dots \dots . +L^{2m+1} + L^m)\ d(x_1, x_0, a) \end{aligned}$$

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$= L^{2m}(1 + L + L^2 + L^3 + \dots\dots.. L^{2n-2m-1})\, d(x_1, x_0, a)$
$\leq \left(\frac{L^{2m}}{1-L}\right).\left(d(x_1, x_0, a)\right)$, as l< 1 and p is closed.

Thus we have

$$|| d(x_{2n}, x_{2m}, a)|| \leq \left(\frac{L^{2m}}{1-L}\right) || d(x_1, x_0, a)||$$

T$his\ implies (d(x_1, x_0, a) \to 0, (n, m \to \infty), for\ all\ a \in X$.

Hence $\{x_n\}$ is a Cauchy sequence. in (X, d) is a complete cone 2 –metric space, then there exist, $u \in X$ such that $x_n \to (n \to \infty.).\lim n \to \infty x_n = u$,

$d(T_1\text{u}, u, a) \leq a_1 d(u, x_{2n}, a) + a_2 d(u, T_1 u, a) + a_3 d(x_{2n}, T x_{2n}, a) + a_4[d(u, T x_{2n}, a) + d(x_{2n}, T_1 u, a)]$
$\leq a_1 d(u, x_{2n}, a) + a_2 d(T_1 u, u, a) + a_3 d(x_{2n}, x_{2n+1}, a)$
$+[a_4 d(u, x_{2n+1}, a) + d(T_1 u, x_{2n}, a)] + d(x_{2n+1}, u, a)$
$\leq a_1 d(u, u, a) + a_2 d(T_1 u, u, a) + a_3[d(u, u, a)] + a_4[\text{d}(u, u, a) + \text{d}(T_1 u, u, a)] + \text{d}(x_{2n+1}, u, a)$

$d(T_1\text{u}, u, a) = (a_2 + a_4) d(T_1 u, u, a)$

On taking limit as $n \to \infty$ and by using 2.6(4), we get $d(T_1 u, u, a) = 0$ this $implies\ T_1 u = u$ .So u is a fixed point of T_1 in X .Now if v is another fixed point of T_1 in , then

$d(u, v, a) = (T_1 u, T_2 v, a)$
$\leq a_1 d(u, v, a) + a_2 d(T_1 u, u, a) + a_3 d(v, v, a) + a_4 d[(v, u, a) + a_5 d(u, v, a)]$
$\leq a_1 d(u, v, a) + a_2 d(u, u, a) + a_3 d(T_2 v, v, a) + a_4 d[(T_2 v, u, a) + a_5 d(T_1 u, v, a)]$
$= (a_1 + 2a_4) d(u, v, a)$

By using remark 2.2 we obtain that $d(u, v, a) = 0$. thus $u = v$. therefore the fixed point of T_1 is unique.

Similarly it can be established that $T_2 u = u$. Hence $T_1 u = u = T_2 u$. Thus u is common fixed point of T_1 and T_2 this completes the proof.

Corollary 3.7: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping$T_1, T_2: X \to X$ satisfy the contractive condition

$d(T_1 x, T_2 y, a) \leq a_1 d(x, y, a) + a_2(x, T_1 x, a)$

for all x, y, a ∈X where $a_1, a_2 \in$ [o,1] are all negative constants a constant with$a_1 + a_2 < 1$. then $T_1$and T_2 have an unique common fixed point in X .and for $any\ x \in X$,iterative sequence $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ is conversant to fixed point.

Proof: putting $a_3 = a_4 = 0$ in theorem 3.6 ,then we get the required results.

Corollary 3.8: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant K. Suppose the mapping T_1 , $T_2: X \to X$ satisfy the contractive condition

$d(T_1 x, T_2 y, a) \leq a_1 d(x, y, a) + a_2(x, T_1 x, a) + a_3 \text{d}(\text{y}, T_2 y, a)$

for all $x, y, a \in X$ where $k \in [0,1]$ is a constant . then then $T_1$and T_2 have an unique common fixed point in X .and for $any\ x \in X$,iterative sequence $\{T_1^{2n+1}x\}$ and$\{T_2^{2n+2}x\}$ is conversant to fixed point.

Proof: Putting $a_4 = 0$ in theorem 3.6 , then we get the required results easily.

Corollary 3.9: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant k. suppose the mapping T_1 , $T_2: X \to X$ satisfy the contractive condition.

$d(T_1 x, T_2 y, a) \leq \alpha[d(x, T_2 y, a) + d(y, T_1 x, a)]$.

$for\ all\ x, y \in X\ \ where\ \alpha \in \left[0, \frac{1}{2}\right].Then\ T_1$ and $T_2\ have\ unique\ fixed\ point\ in\ X.and\ \ for\ all\ x \in X.\ Iterative\ sequence$ $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ is conversant to fixed point.

Proof: Putting $a_1 = a_2 = a_3 = 0$ and $a_4 = \alpha$ theorem 3.6, then we get the required results easily.

Theorems 3.10: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant K. Suppose the mapping T_1 , $T_2: X \to X$ satisfy the contractive condition.

$d(T_1 x, T_2 y, a) \leq a_1 d(x, y, a) + a_2 d(x, T_2 y, a) + a_3 d(y, T_1 x, a) a_4[d(x, T_1 x, a) + d(y, T_2 y, a)]$ (3.10.1)

$for\ all\ x, y \in X. And\ ai\ , i = 1,2,3,4,.Are\ all\ non\ negative\ constant\ with\ a1\ + a2\ + a3\ +\ a4 + <\ 1.then\ T_1$ and $T_2\ have\ unique\ fixed\ point\ in\ X.and\ for\ all\ x \in X.Iterative\ sequence$ $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}\}$ is conversant to fixed point.

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Proof: Let $x_0 \in X$ be fixed point.

$Let\ x_1 = Tx_0, x_3 = T_1 x_1 = T_1^3 x_0 \dots\dots\dots$

$x_{2n+1} = Tx_{2n} = T^{2n+1} x_0 \dots.$

Similarly, we have $x_2 = T_1 x_1, x_3 = T^2 x_0, x_4 = T_2 x_3 = T_2^4 x_0 \dots\dots$

$x_{2n+2} = Tx_{2n+1} = T^{2n+2} x_0$

From (3. 6) Taking $x = x_{2n}$ and y = x_{2n-1}, we have

$$d(x_{2n+1}, x_{2n}, a) = d(T_1 x_{2n}, T_2 x_{2n-1}, a)$$
$$\leq a_1 d(x_{2n}, x_{2n-1,} a) + a_2 d(x_{2n}, T_2 x_{2n-1}, a) + a_3 d(x_{2n-1,} T_1 x_{2n}, a)$$
$$+ a_4 [d(x_{2n}, T_1 x_{2n}, a) + d(x_{2n-1,} T_2 x_{2n-1}, a)]$$
$$\leq a_1 d(x_{2n}, x_{2n-1,} a) + a_2 d(x_{2n}, x_{2n,} a) + a_3 d(x_{2n-1} x_{2n+1}, a)$$
$$+ a_4 [d(x_{2n}, x_{2n+1,} a) + d(x_{2n-1}, x_{2n,} a)]$$
$$\leq (a_1 + a_3 + a_4) d(x_{2n}, x_{2n-1,} a) + (a_3 + a_4)\, d(x_{2n+1,} x_{2n} a)$$
$$(1 - a_3 - a_4)\, d(x_{2n+1,} x_{2n}, a) \leq (a_1 + a_3 + a_4)\, d(x_{2n}, x_{2n-1,} a)$$

$$d(x_{2n+1,} x_{2n} a)) \leq L d(x_{2n}, x_{2n-1,} a)$$

$$Where\ L = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_4} \quad (3.6.2)$$

$$\leq L^2 \{d(x_{2n-1,}, x_{2n-2}, a)\}$$
$$\leq \cdots \dots\dots\dots\dots\dots \leq L^n \{F(x_1, x_0, a)\} (3.6.3)$$

Also for $k > t$, we have

$$d(x_{2k}, x_{2k-1}, x_{2t}) \leq L d(x_{2k-1}, x_{2k-2}, x_{2t})$$
$$\leq L^2 d(x_{2k-2}, x_{2k-3,} x_{2t})$$
$$\leq \cdots \dots\dots\dots\dots\dots\dots\dots\dots. \leq$$
$$\leq L^{2k-2t-1} d(x_{2t+1}, x_{2t}, x_{2t})$$
$$= 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3.6.4)$$

Now for $n > m$, with using (3.63) and (3.6.4), we have

$$d(x_{2n}, x_{2m}, a) \leq d(x_{2n}, x_{2m}, x_{2n-1}) + d(x_{2n}, x_{2n-1}, a) + d(x_{2n-1}, x_{2m}, a)$$
$$\leq L^{2n-1} d(x_1, x_0, a) + d(x_{2n-1}, x_{2m}, x_{2n-2}) + d(x_{2n-1}, x_{2n-2}, a)$$
$$+ d(x_{2n-2}, x_{2n-3}, a) + d(x_{2n-3}, x_{2m}, a)$$
$$\leq (L^{2n-1} + L^{2n-2} + L^{2n-3})\, d(x_1, x_0, a) + d(x_{2n-3}, x_{2m}, a)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\leq (L^{2n-1} + L^{2n-2} + \cdots \dots\dots\dots\dots. + L^{2m+1})\, d(x_1, x_0, a) + d(x_{m+1}, x_m, a)$$
$$\leq (L^{2n-1} + L^{2n-2} + \cdots \dots\dots\dots\dots. + L^{2m+1} + L^m)\, d(x_1, x_0, a)$$
$$= L^{2m} (1 + L + L^2 + L^3 + \dots\dots.. L^{2n-2m-1})\, d(x_1, x_0, a)$$
$$\leq \left(\frac{L^{2m}}{1-L}\right). \left(d(x_1, x_0, a)\right), \text{as l< 1 and p is closed.}$$

Thus we have

$$||\, d(x_{2n}, x_{2m}, a)|| \leq \left(\frac{L^{2m}}{1-L}\right) ||\, d(x_1, x_0, a)||$$

T$his\ implies (d(x_1, x_0, a) \to 0, (n, m \to \infty), for\ all\ a \in X.$

Hence $\{x_n\}$is a Cauchy sequence. in (X, d) is a complete cone 2 –metric space, then there exist, $u \in X$ such that $x_n \to (n \to \infty.). \lim n \to \infty x_n = u,$

$$d(T_1 u, u, a) \leq a_1 d(u, x_{2n}, a) + a_2 d(u, T_1 u, a) + a_3 d(x_{2n}, Tx_{2n}, a) + a_4 [d(u, Tx_{2n}, a) + d(x_{2n}, T_1 u, a)]$$
$$\leq a_1 d(u, x_{2n}, a) + a_2 d(u, x_{2n+1}, a) + a_3 d(x_{2n,} T_1 u, a)$$
$$+ [a_4 d(u, T_1 u, a) + d(x_{2n}, x_{2n+1,} a)] + d(x_{2n+1}, u, a)$$
$$\leq a_1 d(u, u, a) + a_2 d(u, u, a) + a_3 [d(T_1 u, u, a)] + a_4 [d(T_1 u, u, a) + d(u, u, a)] + d(x_{2n+1}, u, a)$$
$$d(T_1 u, u, a) = (a_3 + a_4) d(T_1 u, u, a)$$

On taking limit as $n \to \infty$ and by using 2.6(4), we get $d(T_1 u, u, a) = 0$ this $implies$ $T_1 u = u$. So u is a fixed point of T_1 in X .Now if v is another fixed point of T_1 in, then

$$d(u, v, a) = (T_1 u, T_2 v, a)$$
$$\leq a_1 d(u, v, a) + a_2 d(, u, T_2 v, a) + a_3 d(v, T_1 u, a) + a_4 d[(u, T_1 u, a) + a_5 d(v, T_2 v, a)]$$
$$\leq a_1 d(u, v, a) + a_2 d(u, v, a) + a_3 d(v, u, a) + a_4 d[(u, u, a) + a_5 d(v, v, a)]$$
$$= (a_1 + a_1 + a_3) d(u, v, a)$$

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By using remark 2.2 we obtain that $d(u, v, a) = 0$. thus $u = v$. therefore the fixed point of T_1 is unique.

Similarly it can be established that $T_2 u = u$. Hence $T_1 u = u = T_2 u$. Thus u is common fixed point of T_1 and T_2 this completes the proof.

Corollary 3.11: Let (X, d) be a complete cone metric space. P be a normal cone with normal constant K. Suppose the mapping T_1 , $T_2 : X \rightarrow X$ satisfy the contractive condition.

$$d(T_1 x, T_2 y, a) \leq a_1 d(x, y, a) + a_2 d(x, T_2 y, a) + a_3 d(y, T_1 x, a)$$

$for\ all\ x, y \in X. And\ ai\ , i = 1,2,3 \in [0,1] Are\ all\ non\ negative\ constant\ with\ a_1 + a_2 + a_3 + < 1.\ then\ T_1$ and $T_2\ have\ unique\ common\ fixed\ point\ in\ X. and\ for\ all\ x \in X. Iterative\ sequence$ $\{T_1^{2n+1} x\}$ and $\{T_2^{2n+2} x\}\}$ is conversant to fixed point.

Proof: Putting and $a_4 = 0$. theorem 3.6, then we get the required results easily.

4. CONCLUSION

Theorem concerning the existence and uniqueness of solution of cone 2 – metric space and established and improve some Banach contraction theorems for contractive mappings of the results of S.K. Tiwari *et al.* [17].

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